ARML Competition 2016

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1 Team Problems

Problem 1. Compute all values of a for which the intersection of the graphs of $y \geq \left|\frac{x}{2}\right|$ and $y \leq a|x| + 17$ is a region having area 51.

Problem 2. Let $f(x) = \log_b x$ and let $g(x) = x^2 - 4x + 4$. Given that $f(g(x)) = g(f(x)) = 0$ has exactly one solution and that $b > 1$, compute b.

Problem 3. Compute all solutions to the following system of equations:

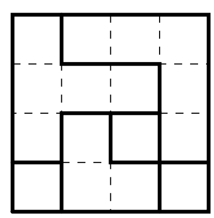
$$\begin{aligned} |y| - \frac{2x}{|x|} &= -1 \\ x|x| + y|y| &= 24. \end{aligned}$$

Problem 4. Two players are playing a game as follows. Integers N and M are each selected independently at random from the set $\{1, 2, \ldots, 50\}$. The players begin with an $N \times M$ grid of empty squares. A turn consists of filling in a single empty square, then filling in all the squares in the same row, and then filling in all the squares in the same column. The players alternate taking turns until all squares are filled in. Compute the probability that the player who goes first fills in the last square.

Problem 5. A sequence $T_1, T_2, \ldots$ is called (a, b)-*nacci* if $T_1 = a$, $T_2 = b$, and for $n \geq 3$, $T_n = T_{n-1} + T_{n-2}$. Compute the number of (a, b)-nacci sequences of positive integers such that $b > a$ and there exists an integer $k \geq 3$ with $T_k = 50$.

Problem 6. Complete the number puzzle below. Clues are given for the four rows. (Answers may not begin with a zero.) Cells inside a region must all contain the same digit, and each region contains a different digit.

1. Product of two primes
2. Multiple of 19
3. A perfect square
4. Multiple of 1-Across



Problem 7. Real numbers x, y, and z are chosen at random from the unit interval $[0, 1]$. Compute the probability that $\max\{x, y, z\} - \min\{x, y, z\} \leq \frac{2}{3}$.

Problem 8. Square $ARML$ has sides of length 11. Collinear points X, Y, and Z are in the interior of $ARML$ such that $\triangle LXY$ and $\triangle RXZ$ are equilateral. Given that $YZ = 2$, compute $[RLX]$.

Problem 9. Compute the number of permutations $x_1, x_2, \ldots, x_{10}$ of the integers $-3, -2, -1, \ldots, 6$ that satisfy the chain of inequalities

$$x_1x_2 \leq x_2x_3 \leq \cdots \leq x_9x_{10}.$$

Problem 10. Compute the greatest integer n for which the decimal representation of $\frac{1000}{n}$ is of the form $\underline{1}\,.\,\underline{T}\,\underline{A}\,\underline{S}\,\underline{T}\,\underline{Y}\,\underline{7}\ldots$, where the digits T, A, S, and Y are not necessarily distinct.

2 Answers to Team Problems

Answer 1. $-\frac{31}{6}$

Answer 2. $\sqrt{3}$

Answer 3. $(5, -1)$ and $(\sqrt{23}, 1)$

Answer 4. $\frac{51}{100}$

Answer 5. 37

Answer 6.

4	6	6	6
4	4	4	6
4	3	5	6
9	3	3	2

Answer 7. $\frac{20}{27}$

Answer 8. $\frac{119\sqrt{3}}{6}$

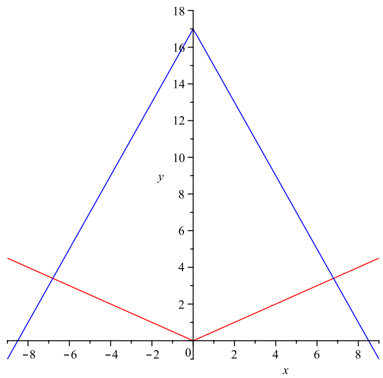
Answer 9. 240

Answer 10. 968

3 Solutions to Team Problems

Problem 1. Compute all values of a for which the intersection of the graphs of $y \geq \left|\frac{x}{2}\right|$ and $y \leq a|x| + 17$ is a region having area 51.

Solution 1. For $a < 0$, the graphs of the two functions are shown in the figure below.



In the first quadrant, the two bounding functions are $y = \frac{x}{2}$ and $y = ax+17$, so they intersect where $\frac{x}{2} = ax+17$. Moreover, because the triangle in the first quadrant is half of the total area it has area $\frac{51}{2}$. Taking the side along the y-axis as the base, the height is the x-value where the two lines intersect, hence $\frac{1}{2} \cdot 17 \cdot x = \frac{51}{2}$, thus $x = 3$. Now $\frac{3}{2} = 3a + 17$, so $a = -\boldsymbol{\frac{31}{6}}$.

Problem 2. Let $f(x) = \log_b x$ and let $g(x) = x^2 - 4x + 4$. Given that $f(g(x)) = g(f(x)) = 0$ has exactly one solution and that $b > 1$, compute b.

Solution 2. Consider the equation $f(g(x)) = 0$. Because $f(x) = \log_b(x)$, this equation holds precisely when $g(x) = 1$, no matter what b is. Setting $g(x) = x^2-4x+4$ equal to 1 yields the quadratic equation $x^2-4x+3 = 0$, which has two solutions, $x = 3$ and $x = 1$. Now consider the equation $g(f(x)) = 0$: $g(x)$ has one root at 2, so $f(x) = 2$, which implies that $x = b^2$. Therefore, in order for the equation $f(g(x)) = g(f(x)) = 0$ to have exactly one solution, b must equal $\boldsymbol{\sqrt{3}}$.

Problem 3. Compute all solutions to the following system of equations:

$$\begin{aligned} |y| - \frac{2x}{|x|} &= -1 \\ x|x| + y|y| &= 24. \end{aligned}$$

Solution 3. Note that x must be real because otherwise, the first equation would not be satisfied.

Case 1: x is positive. Then $\frac{2x}{|x|} = 2$, and the first equation yields $y = \pm 1$. The second equation then becomes $x^2 \pm 1 = 24$, so $x = 5$ (when $y = -1$) or $x = \sqrt{23}$ (when $y = 1$).

Case 2: x is negative. Then $\frac{2x}{|x|} = -2$, but in this case, the first equation becomes $|y| = -3$, which is impossible.

Thus the two solutions are $\boldsymbol{(5, -1)}$ and $\boldsymbol{(\sqrt{23}, 1)}$.

Problem 4. Two players are playing a game as follows. Integers N and M are each selected independently at random from the set $\{1, 2, \ldots, 50\}$. The players begin with an $N \times M$ grid of empty squares. A turn consists of filling in a single empty square, then filling in all the squares in the same row, and then filling in all the squares in the same column. The players alternate taking turns until all squares are filled in. Compute the probability that the player who goes first fills in the last square.

Solution 4. The outcome of the game is determined by the parity of $\min(N, M)$. This is because each time a player takes a turn, a row and a column that are previously not completely filled in (i.e., the row and column that contain the square selected on that turn) are now both completely filled in. Therefore, as soon as $\min(N, M)$ turns are taken, either all the rows will be completely filled, or all the columns, which means that the entire grid is filled in. Therefore the first player will fill in the last square if and only if $\min(N, M)$ is odd.

To compute the probability than $\min(N, M)$ is odd, count the number of pairs (N, M) satisfying this condition, and then divide by $50 \times 50 = 2500$, the total number of possible choices of N and M. First consider the case when $N < M$. There are 25 choices for N, from 1 to 49, and each results in $50 - N$ choices for M (i.e., the number of numbers larger than N). So there are $49 + 47 + \ldots + 3 + 1 = 25^2 = 625$ such pairs (N, M). The case when $M < N$ is symmetric, so there are another 625 cases there. Then there are another 25 cases where $N = M$, because N and M can equal any odd number between 1 and 50. This is a total of $625+625+25 = 1275$ pairs, so the desired probability is $\frac{1275}{2500} = \mathbf{\frac{51}{100}}$.

Problem 5. A sequence $T_1, T_2, \ldots$ is called (a, b)-*nacci* if $T_1 = a$, $T_2 = b$, and for $n \geq 3$, $T_n = T_{n-1} + T_{n-2}$. Compute the number of (a, b)-nacci sequences of positive integers such that $b > a$ and there exists an integer $k \geq 3$ with $T_k = 50$.

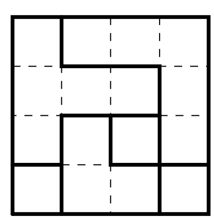
Solution 5. The sequence T_k must grow at least as fast as the $(1, 2)$-nacci sequence $1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$, thus for $k \geq 9$, $T_k > 50$.

If $k = 3$, then $a + b = 50$, which has 24 solutions with a being any integer from 1 to 24 inclusive.
If $k = 4$, then $a + 2b = 50$. Now a must be even and $a \leq 16$, so there are 8 solutions for $k = 4$.
If $k = 5$, then $2a + 3b = 50$. Now $a \equiv 1 \pmod 3$ and $a < 10$, so there are 3 solutions for $k = 5$.
If $k = 6$, then $3a + 5b = 50$. Now a must be a multiple of 5 and the only solution is $(5, 7)$.
If $k = 7$, then $5a + 8b = 50$. Now b must be a multiple of 5 and the only solution is $(2, 5)$.
If $k = 8$, then $8a + 13b = 50$, which has no solutions with $a < b$.

Thus the desired number of (a, b)-nacci sequences is $24 + 8 + 3 + 1 + 1 = \mathbf{37}$.

Problem 6. Complete the number puzzle below. Clues are given for the four rows. (Answers may not begin with a zero.) Cells inside a region must all contain the same digit, and each region contains a different digit.

1. Product of two primes
2. Multiple of 19
3. A perfect square
4. Multiple of 1-Across



Solution 6. Consider first 1-Across, which is of the form $\underline{A}\,\underline{B}\,\underline{B}\,\underline{B}$. 4-Across is a multiple of 1-Across, and of a different form, so they are not equal. Because both are four-digit numbers, A must be less than 5.

Next look at 2-Across. It is a four-digit multiple of 19 of the form $\underline{A}\,\underline{A}\,\underline{A}\,\underline{B}$, where A and B are the same digits as above. Therefore 2-Across equals $1110A + B$, which is equivalent to $8A + B$ mod 19. Because 2-Across is a multiple of 19 and $A < 5$, the possibilities for (A, B) are $(2, 3)$ and $(4, 6)$.

Therefore 1-Across is either 2333 or 4666. The four-digit multiples of 2333 are 4666, 6999, and 9332. Only 9332 fits the pattern of 4-Across, so 4-Across is 9332. Because the digits in each region are distinct, 1-Across must be 4666.

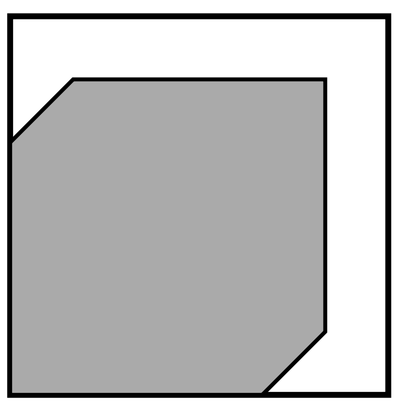
Then 2-Across is 4446. Filling in the digits that are the same in each region, 3-Across must be $43\underline{X}6$, where X is 0, 1, 5, 7, or 8, because these are the only unused digits. Noting that 3-Across is a perfect square and that $4356 = 66^2$, 3-Across is 4356.

4	6	6	6
4	4	4	6
4	3	5	6
9	3	3	2

Problem 7. Real numbers x, y, and z are chosen at random from the unit interval $[0, 1]$. Compute the probability that $\max\{x, y, z\} - \min\{x, y, z\} \leq \frac{2}{3}$.

Solution 7. Consider points lying in a unit cube in the first octant with one corner at the origin. To find the volume of the region of points where the inequality holds, take cross-sections for each value of z from 0 to 1. For any particular value of z, the values of x and y must satisfy $0 \leq x \leq 1$, $0 \leq y \leq 1$, $-\frac{2}{3} \leq y - x \leq \frac{2}{3}$, $z - \frac{2}{3} \leq x \leq z + \frac{2}{3}$, and $z - \frac{2}{3} \leq y \leq z + \frac{2}{3}$.

Depending on the value of z, the resulting region is a square of area $\frac{4}{9}$ (when $z = 0$ or 1), a hexagon of area $\frac{4}{9} + \frac{4z}{3}$ (when $0 < z < \frac{1}{3}$), a hexagon of area $\frac{8}{9}$ (when $\frac{1}{3} \leq z \leq \frac{2}{3}$), or a hexagon of area $\frac{4}{9} + \frac{4(1-z)}{3}$ (when $\frac{2}{3} < z < 1$). The region where $z = \frac{1}{6}$ is shown.



In the cases where z is not between $\frac{1}{3}$ and $\frac{2}{3}$, the area changes linearly with z and hence can be averaged to $\frac{1}{2}\left(\frac{4}{9} + \frac{8}{9}\right) = \frac{2}{3}$. This area corresponds to $\frac{2}{3}$ of the z-values, with the other $\frac{1}{3}$ having area $\frac{8}{9}$, for a total probability of $\frac{2}{3}(\frac{2}{3}) + \frac{1}{3}(\frac{8}{9}) = \mathbf{\frac{20}{27}}$.

Problem 8. Square $ARML$ has sides of length 11. Collinear points X, Y, and Z are in the interior of $ARML$ such that $\triangle LXY$ and $\triangle RXZ$ are equilateral. Given that $YZ = 2$, compute $[RLX]$.

Solution 8. First note that X cannot be between Y and Z, because then the side lengths of triangles LXY and RXZ would each be less than $YZ = 2$, which would mean that XL and XR would each be less than 2, which is impossible because $XL + XR \geq LR = 11\sqrt{2}$. Without loss of generality, let Y lie between X and Z. Let $XY = YL = LX = x$, so that $XZ = ZR = RX = x + 2$. In triangle RLX, angle LXR has measure 120° because angles LXY and ZXR are each 60°. Applying the Law of Cosines then gives

$$\frac{x^2 + (x+2)^2 - (11\sqrt{2})^2}{2x(x+2)} = \cos 120^\circ = -\frac{1}{2}.$$

Solving this quadratic for x gives $x = -1 \pm \sqrt{\frac{241}{3}}$, and the positive solution for x is used because it is a side length. So $x = -1 + \sqrt{\frac{241}{3}}$ and $x + 2 = 1 + \sqrt{\frac{241}{3}}$, and the area $[RLX]$ is

$$\frac{1}{2}x(x+2)\sin 120^\circ = \frac{1}{2} \cdot \left(\frac{241}{3} - 1\right) \cdot \frac{\sqrt{3}}{2} = \frac{\mathbf{119\sqrt{3}}}{\mathbf{6}}.$$

Problem 9. Compute the number of permutations $x_1, x_2, \ldots, x_{10}$ of the integers $-3, -2, -1, \ldots, 6$ that satisfy the chain of inequalities

$$x_1x_2 \leq x_2x_3 \leq \cdots \leq x_9x_{10}.$$

Solution 9. First note that no two negative numbers can occur consecutively because otherwise, either the resulting positive product would have to be followed by a non-positive product once the negatives are exhausted, or it would have to occur at the end but there are too many positive terms for that to be possible. Thus the sequence must start with alternating positive and negative values. Each inequality $x_ix_{i+1} \leq x_{i+1}x_{i+2}$ implies that $x_i < x_{i+2}$ if $x_{i+1} > 0$ and $x_i > x_{i+2}$ if $x_{i+1} < 0$ (the inequalities are strict because the x_is are distinct). The two main cases to consider are $x_1 = -3$ and $x_2 = -3$.

If $x_1 = -3$, then $x_3 = -2$, $x_5 = -1$, and either $x_6 = 0$ or $x_7 = 0$. If $x_6 = 0$, then the positive entries (x_2, x_4, x_7, x_8, x_9, and x_{10}) must satisfy $x_2 > x_4$, $x_7 < x_9$, and $x_8 < x_{10}$. There are $\frac{6!}{2! \cdot 2! \cdot 2!} = 90$ such sequences. If $x_7 = 0$, then the positive entries (x_2, x_4, x_6, x_7, x_8, and x_{10}) must satisfy $x_2 > x_4 > x_6$, and $x_8 < x_{10}$ for $\frac{6!}{3! \cdot 2! \cdot 1!} = 60$ more.

If $x_2 = -3$, then $x_4 = -2$, $x_6 = -1$, and either $x_7 = 0$ or $x_8 = 0$. If $x_7 = 0$, then the positive entries (x_1, x_3, x_4, x_8, x_9, and x_{10}) must satisfy $x_1 > x_3 > x_5$, and $x_8 < x_{10}$ for another 60. If $x_8 = 0$, then the positive entries (x_1, x_3, x_5, x_7, x_9, and x_{10}) must satisfy $x_1 > x_3 > x_5 > x_7$, for $\frac{6!}{4! \cdot 1! \cdot 1!} = 30$ more.

The desired number of permutations is therefore $90 + 60 + 60 + 30 = \mathbf{240}$.

Problem 10. Compute the greatest integer n for which the decimal representation of $\frac{1000}{n}$ is of the form $\underline{1}.\underline{T}\,\underline{A}\,\underline{S}\,\underline{T}\,\underline{Y}\,\underline{7}\ldots$, where the digits T, A, S, and Y are not necessarily distinct.

Solution 10. In order for the integral part of $\frac{1000}{n}$ to be 1, n must be between 501 and 1000. Let $m = 1000 - n$. Then

$$\frac{1000}{n} = \frac{1000}{1000 - m} = \frac{1}{1 - \frac{m}{1000}} = 1 + \frac{m}{1000} + \frac{m^2}{1000^2} + \frac{m^3}{1000^3} + \ldots.$$

For example, if $n = 998$, then $\frac{1000}{n}$ begins $1.002004008016032064\ldots$ and each block of 3 digits following the decimal point is a power of 2 until they begin to overlap.

To maximize n, first consider attempting to find a solution with $T = 0$, which implies $m < 100$, by the previous analysis, because the first block of 3 digits equals either m or m plus the thousands digit of m^2. Because the ones digit of the second block of 3 digits is 7, and m^2 cannot end in 7 (as no squares end in 7), the m^3 block must overlap into the m^2 block. Therefore $m^3 > 1000$, so $m > 10$.

Because $T = 0$ by (optimistic) assumption, and therefore the part of m^2 in the second block must have a zero in the hundreds place, the goal is to find four-digit squares with a zero in the hundreds place. (Note that m cannot have a two-digit square because $m > 10$, and the m^3 block will not overlap onto the 0 from the hundreds place of the second block as long as $m^3 < 100{,}000$.)

The first square with a zero in the hundreds place is $32^2 = 1024$, which corresponds to $m = 32$, $m^2 = 1024$, $m^3 = 32768$. Then the expansion for $\frac{1000}{1000-m}$ begins

$$\begin{aligned} & 1 + \frac{32}{1000} + \frac{1024}{1000^2} + \frac{32768}{1000^3} + \cdots \\ = \ & 1 + 0.032 + 0.001024 + 0.000032768 + \cdots \\ = \ & 1.033056768 + \cdots . \end{aligned}$$

This fits the pattern, except the sixth decimal digit is a 6 instead of a 7. However, $m^4 = 1024^2$ is just over 1 million, so its first digit overlaps onto the 6, making it a 7. Therefore $m = 32$, which corresponds to $n = \mathbf{968}$ giving the maximal n satisfying the given criteria. This can be confirmed by computing $\frac{1000}{968} = 1.033057\ldots$.

Note: The same method as in the solution can be used for negative integers m that are close to 0; the geometric series would have alternating signs. Also, replacing the 1000s with 10^ps yields a similar result, but instead of blocks of 3 digits apiece, the blocks will each have p digits.

4 Power Question 2016: Taxicab Geometry

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

In standard coordinate (Euclidean) geometry, if $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then the distance between A and B is given by the Euclidean distance function: $d_E(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. In this Power Question, we investigate what happens when distance is measured using the *taxicab* distance function,

$$d_T(A, B) = |x_2 - x_1| + |y_2 - y_1|.$$

This distance is called the *taxi-distance* between A and B.

1. This problem will help you become familiar with taxi-distances.

a. Compute the taxi-distance between the points $(2, 6)$ and $(7, -2)$. [1 pt]

b. Compute the coordinates of a point whose Euclidean distance from $(3, 2)$ is 5 and whose taxi-distance from $(3, 2)$ is also 5. [1 pt]

c. Compute the coordinates of a point whose Euclidean distance from $(3, 2)$ is 5 and whose taxi-distance from $(3, 2)$ is 7. [1 pt]

In Euclidean geometry, one possible definition of the line segment $\overline{AB}$ is the set of points P such that $AP + PB = AB$, or $d_E(A, P) + d_E(P, B) = d_E(A, B)$. This definition does not work as well in taxicab geometry!

2. Let $A = (0, 4)$ and $B = (4, 0)$.

a. Show that $P = (1, 3)$ satisfies $d_T(A, P) + d_T(P, B) = d_T(A, B)$. [1 pt]

b. Find the coordinates of a point P *not* on the line $x + y = 4$ satisfying $d_T(A, P) + d_T(P, B) = d_T(A, B)$. [1 pt]

c. Find the set of all points P such that $d_T(A, P) + d_T(P, B) = d_T(A, B)$, and justify your answer. [3 pts]

As in Euclidean geometry, two triangles are congruent if and only if the lengths of corresponding sides and the measures of corresponding angles are equal. In Euclidean geometry, this *measure-congruence* is equivalent to *transformation-congruence*: two triangles are transformation-congruent if and only if one is the image of the other under a composition of reflections, rotations, and translations. (Actually, reflections alone suffice, but that result is beyond the scope of this Power Question.) For the remainder of this Power Question, the symbol $\cong$ will be used exclusively to denote measure-congruence; we write $\cong_T$ to emphasize that we are using taxicab measure.

3. Let's investigate whether the equivalence between measure- and transformation-congruence holds in taxicab geometry. Suppose that $A = (x_A, y_A)$ and similarly for other named points.

a. If $\triangle A'B'C'$ is the image of $\triangle ABC$ under the translation $(x, y) \mapsto (x + h, y + k)$, are the taxi-lengths of the sides of $\triangle A'B'C'$ equal to those of $\triangle ABC$? Justify your answer. [1 pt]

b. If $\triangle A'B'C'$ is the image of $\triangle ABC$ under a reflection in the line $y = k$, are the taxi-lengths of the sides of $\triangle A'B'C'$ equal to those of $\triangle ABC$? Justify your answer. [1 pt]

c. Let $O = (0, 0)$, $R = (8, 0)$, $D = (0, 4)$, and let $\triangle O'R'D'$ be the image of $\triangle ORD$ under a (Euclidean) rotation of $45°$ counterclockwise about the origin. Is $\triangle ORD \cong_T \triangle O'R'D'$? Justify your answer. [2 pts]

4. Let O have coordinates $(0,0)$, let P have coordinates $(1,0)$, and let A be the point $(4,2)$. The point B has coordinates (m,n), where m and n are both positive integers, m is relatively prime to n, and $\angle BOA \cong \angle AOP$ (considered as Euclidean angles).

- **a.** Compute the ordered pair (m,n). [2 pts]
- **b.** Compute the taxi-lengths of the segments intercepted by $\angle BOA$ and $\angle AOP$ on the line $x+y=1$. [2 pts]

The computation in the previous item is problematic, for the following reason. A *taxi-circle* is the set of all points that are a fixed taxi-distance from a given point, and the segments whose lengths you computed in 4b are actually arcs on a unit taxi-circle centered at the origin. So angles congruent in the Euclidean sense do not necessarily intercept taxi-congruent arcs.

One way to fix this problem is to define the measure of an angle analogously to the way the radian measure of an angle is defined in Euclidean geometry, namely, as a quotient. The numerator is the length of the arc that the angle intercepts on a circle centered at the angle's vertex. The denominator is the radius of that circle. If P and R are equidistant from Q, then the *measure of $\angle PQR$ in taxi-radians* is given by the formula

$$\mathrm{m}_T\angle PQR = \frac{\text{taxi-length of the arc from } P \text{ to } R \text{ on the taxi-circle centered at } Q \text{ passing through } P}{d_T(Q,P)}.$$

5. Let's flesh out the above definition with an example.

- **a.** Find an equation to describe the taxi-circle centered at the origin of radius 17. [1 pt]
- **b.** Compute the value of *taxi-pi*, π_T, by dividing the semi-taxi-circumference of the taxi-circle from part 5a by its radius. [1 pt]
- **c.** Show that the value you computed in part 5b is independent of the taxi-circle's radius. [2 pts]
- **d.** Given that $C=(3,1)$, $O=(0,0)$, and $B=(-1,3)$, compute $\mathrm{m}_T\angle COB$. [1 pt]

Using taxi-circles and taxi-angles, define a *rotation by θ taxi-radians about the origin* as follows. The point P' is the image of P under a counterclockwise rotation of θ taxi-radians around $O=(0,0)$ if and only if $d_T(O,P)=d_T(O,P')$ and the measure of the *taxi-arc* from P to P' (going counterclockwise around the taxi-circle centered at O) is $d_T(O,P)\cdot\theta$.

6. Let $O=(0,0)$, $A=(8,0)$, and $R=(3,1)$.

- **a.** Let $\triangle O'A'R'$ be the image of $\triangle OAR$ under a rotation of 1 taxi-radian about the origin. Compute the coordinates of O', A', and R'. [3 pts]
- **b.** Is $\triangle O'A'R' \cong_T \triangle OAR$? Justify your answer. [3 pts]

7. We now consider conditions for two triangles to be measure-congruent in Euclidean geometry and apply them to taxicab geometry to discover whether or not measure-congruence still holds. Let $\triangle MOD$ have vertices $M=(0,2)$, $O=(0,0)$, and $D=(2,0)$.

- **a.** Show that there exists a triangle $\triangle CAB$ such that $d_T(C,A)=d_T(M,O)$, $d_T(A,B)=d_T(O,D)$, and $\mathrm{m}_T\angle CAB=\mathrm{m}_T\angle MOD$, but $\triangle MOD \not\cong_T \triangle CAB$. (Thus SAS congruence does not hold in taxicab geometry.) [2 pts]
- **b.** Show that there exists a triangle $\triangle PIG$ such that $d_T(P,I)=d_T(M,O)$, $d_T(I,G)=d_T(O,D)$, and $d_T(P,G)=d_T(M,D)$, but $\triangle MOD \not\cong_T \triangle PIG$. (Thus SSS congruence does not hold in taxicab geometry.) [2 pts]

8. Define the *taxi-cosine* and *taxi-sine* of an angle as follows:

$$\begin{aligned}\cos_T\angle ACB &= \frac{d_E(C,A)\cdot d_E(C,B)}{d_T(C,A)\cdot d_T(C,B)}\cos\angle ACB\\ \sin_T\angle ACB &= \frac{d_E(C,A)\cdot d_E(C,B)}{d_T(C,A)\cdot d_T(C,B)}\sin\angle ACB,\end{aligned}$$

where $\cos\angle ACB$ and $\sin\angle ACB$ are the ordinary (Euclidean) cosine and sine functions, respectively.

a. Given $A = (3, 0)$, $O = (0, 0)$, and $B = (3, 4)$, compute $\cos_T \angle AOB$. [1 pt]

b. Show that if $\overrightarrow{OA}$ is parallel to the x-axis, then $\cos_T \angle AOB = \dfrac{\cos \angle AOB}{|\cos \angle AOB| + |\sin \angle AOB|}$. [3 pts]

c. Let $\overrightarrow{QP}$ be parallel to the x-axis, and let A and B be points not on $\overrightarrow{QP}$. Show that

$$\cos_T \angle AQB = \frac{\cos \angle AQB}{(|\cos \angle AQP| + |\sin \angle AQP|)(|\cos \angle BQP| + |\sin \angle BQP|)}.$$

[3 pts]

9. Suppose that triangle ABC is given with $\overline{AB}$ parallel to the x-axis, and that the perpendicular dropped from C to $\overleftrightarrow{AB}$ intersects $\overline{AB}$ at C' and has taxi-length h_T. As with conventional trigonometry, let $a_T = d_T(B, C)$, $b_T = d_T(A, C)$, and $c_T = d_T(A, B)$. Prove the following:

a. $a_T = b_T + c_T - 2b_T \cos_T A$. [3 pts]

b. $b_T = a_T + c_T - 2a_T \cos_T B$. [2 pts]

c. $c_T = \dfrac{a_T^2 + b_T^2 - 2a_T b_T \cos_T C}{a_T + b_T}$. [3 pts]

10. Given $\triangle ABC$, prove that $\dfrac{a_T}{\sin_T A} = \dfrac{b_T}{\sin_T B} = \dfrac{c_T}{\sin_T C}$, where a_T, b_T, and c_T are defined as in the previous problem. [4 pts]

5 Solutions to Power Question

1. Compute from the given definition:

a. $|7-2|+|-2-6|=13$.

b. There are four points that can found by moving from $(3,2)$ along vectors parallel to the coordinate axes. These points are $(8,2),(-2,2),(3,7)$, and $(3,-3)$. *One of these is sufficient to answer the question.*

c. The classic 3−4−5 right triangle gives many more possibilities: $(3\pm 3, 2\pm 4)$ and $(3\pm 4, 2\pm 3)$. (One of these is sufficient to answer the question.)

2. **a.** $d_T(A,P)+d_T(P,B)=|1-0|+|3-4|+|4-1|+|0-3|=8=|4-0|+|0-4|=d_T(A,B)$.

b. Many solutions exist (see part c below), but $(2,4)$ is one such point.

c. Because $d_T(A,B)=8$, to move from A to B, find all points $P(x,y)$ such that $|x-0|+|y-4|+|4-x|+|0-y|=8$. If $0\le x\le 4$ and $0\le y\le 4$, the left hand side of the equation simplifies to $x+(4-y)+(4-x)+y$, which equals 8 for all such x and y. On the other hand, if $x<0$ and $0\le y\le 4$, then the left hand side of the equation simplifies to $4-2x+(4-y)+y=8-2x$, and for $x<0$, $8-2x>8$. Analogous arguments show that no points with $x>4$, or $y<0$, or $y>4$ satisfy the condition. Hence the solution set is the square $0\le x\le 4, 0\le y\le 4$.

Alternate solution: Let $P=(x,y)$, so that $d_T(A,P)+d_T(P,B)=|x-0|+|y-4|+|4-x|+|0-y|=|4-x|+|x-0|+|4-y|+|y-0|$. For any real number r, $|r|\ge r$, and equality holds if and only if $r\ge 0$. Applying this observation to each term in the above sum, $d_T(A,P)+d_T(P,B)\ge(4-x)+(x-0)+(4-y)+(y-0)=8=d_T(A,B)$, with equality if and only if each of $4-x$, $x-0$, $4-y$, $y-0$ is nonnegative. Hence the solution set is the square $0\le x\le 4$ and $0\le y\le 4$.

3. **a.** Let $A=(x_1,y_1)$ and let $B=(x_2,y_2)$. Then $d_T(A,B)=|x_2-x_1|+|y_2-y_1|$. On the other hand, $A'=(x_1+h,y_1+k)$ and $B'=(x_2+h,y_2+k)$, yielding $d_T(A',B')=|(x_2+h)-(x_1+h)|+|(y_2+k)-(y_1+k)|=|x_2-x_1|+|y_2-y_1|=d_T(A,B)$. The same argument would apply to the taxi-lengths of $\overline{BC}$ and $\overline{AC}$. Therefore the taxi-lengths of the sides are preserved under translation.

b. Let $A=(x_1,y_1)$ and let $B=(x_2,y_2)$. The reflection takes $(x,y)=(x,k+(y-k))$ to $(x,k-(y-k))=(x,2k-y)$. Hence $A'=(x_1,2k-y_1)$ and $B'=(x_2,2k-y_2)$. Thus $d_T(A',B')=|x_2-x_1|+|(2k-y_2)-(2k-y_1)|=|x_2-x_1|+|y_1-y_2|=d_T(A,B)$. Analogous arguments apply to the taxi-lengths of $\overline{BC}$ and $\overline{AC}$, so the taxi-lengths of the sides are preserved under reflection in a horizontal line. (Note: It would be straightforward to show the same result for reflection in a vertical line. What might be difficult about reflecting in an oblique line?)

c. $O'=(0,0)$, $R'=(4\sqrt{2},4\sqrt{2})$, $D'=(-2\sqrt{2},2\sqrt{2})$. Notice that $d_T(O,D)=4$, $d_T(D,R)=12$, and $d_T(O,R)=8$, but $d_T(O',R')=8\sqrt{2}$. Therefore $\triangle ORD\not\cong_T\triangle O'R'D'$.

4. **a.** Let θ be the angle between the positive x-axis and the ray $\overrightarrow{OA}$. Then $\tan\theta=\frac{1}{2}$. The ray $\overrightarrow{OB}$ makes an angle of 2θ with the positive x-axis, so using the double-angle identity from trigonometry yields $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{1}{1-\left(\frac{1}{2}\right)^2}=\dfrac{4}{3}$. Hence B lies on the line $y=\frac{4}{3}x$. Because m and n can have no prime factors in common, $m=3$ and $n=4$. The desired point is $(3,4)$.

b. The rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ intersect the line $x+y=1$ at $\left(\frac{2}{3},\frac{1}{3}\right)$ and $\left(\frac{3}{7},\frac{4}{7}\right)$ respectively. Note that $d_T\left(P,\left(\frac{2}{3},\frac{1}{3}\right)\right)=|1-\frac{2}{3}|+|0-\frac{1}{3}|=\frac{2}{3}$. However, $d_T\left(\left(\frac{2}{3},\frac{1}{3}\right),\left(\frac{3}{7},\frac{4}{7}\right)\right)=|\frac{3}{7}-\frac{2}{3}|+|\frac{4}{7}-\frac{1}{3}|=\frac{10}{21}$. Therefore central angles with the same Euclidean angle measure can intercept arcs of different taxi-lengths on the unit taxi-circle.

5. **a.** The taxi-circle is the set of all points (x,y) such that $|x|+|y|=17$. This equation describes a square centered at the origin with vertices $(\pm 17,0)$ and $(0,\pm 17)$; the sides of the square lie on the lines $x+y=17$, $x+y=-17$, $x-y=17$, and $x-y=-17$.

b. The semi-taxi-circumference of this circle is 68, because $d_T((17,0),(0,17))=d_T((0,17),(-17,0))=34$. (Note that the distance along the arc is, as usual, longer than the straight-line taxi-distance $d_T((17,0),(-17,0))$, which is simply 34.) The radius of the circle is 17. So $\pi_T=68/17=4$.

c. In general, the taxi-circle of radius r centered at the origin is given by the equation $|x| + |y| = r$, and one semicircle has endpoints $(r, 0)$ and $(-r, 0)$. Hence the arc length is simply $d_T((r,0),(0,r)) + d_T((0,r),(-r,0)) = 2r + 2r = 4r$. Therefore $\pi_T = 4r/r = 4$. (As an interesting exercise, you might try to show that this result holds for *any* taxi-semicircle with endpoints (x, y) and $(-x, -y)$.)

d. These points are both found on a taxi-circle of radius 4.

$$\mathrm{m}_T\angle COB = \frac{\text{taxi-length of the arc from } C \text{ to } B \text{ on taxi-circle } O}{d_T(O,C)} = \frac{8}{4} = 2 \text{ taxi-radians.}$$

6. **a**. First note that $O' = O$ because O is the center of rotation. To find A', note that $d_T(O, A) = 8$, so the arc from A to A' on the circle centered at O passing through A must also have length 8, and if A' is in the first quadrant, then it lies on the line $x + y = 8$. These conditions are satisfied by the point $(4, 4)$. Similarly, R' must lie on the line $x + y = 4$ and $d_T(R, R') = 4$. Assuming that $0 < x < 3$ and $1 < y < 4$ yields $d_T(R, R') = (3 - x) + (y - 1) = y - x + 2$. Hence the desired point satisfies both $x + y = 4$ and $y - x + 2 = 4$, yielding $x = 1, y = 3$ as the unique solution. Thus $R' = (1, 3)$.

Alternate solution: First note that $O' = O$ because O is the center of rotation. Before considering the other points, let $P = (x, y)$ be any point in the first quadrant and let $0 \le r \le x$. Then $Q = (x - r, y + r)$ is also in the first quadrant, Q and P are on the same taxi-circle centered at O, and $d_T(P, Q) = 2r$. Apply this observation to $A = (8, 0)$ with $r = \frac{1}{2}d_T(O, A) = 4$ to find $A' = (8-4, 0+4) = (4, 4)$. Similarly, take $P = B = (3, 1)$ and $r = \frac{1}{2}d_T(O, B) = 2$ to find $B' = (3 - 2, 1 + 2) = (1, 3)$.

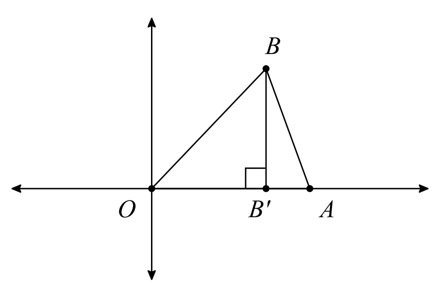
b. By construction, $d_T(O, A) = d_T(O, A')$ and $d_T(O, B) = d_T(O, B')$. However, $d_T(A, B) = |3-8|+|1-0| = 6$ while $d_T(A', B') = |1 - 4| + |3 - 4| = 4$. Hence the two triangles are *not* measure-congruent.

7. **a**. There are many examples. One such example is $\triangle CAB$ with vertices $C(-1, 1)$, $A(0, 0)$, and $B(1, 1)$. Then $d_T(C, A) = d_T(M, O) = 2$, $d_T(A, B) = d_T(O, D) = 2$, and $\mathrm{m}_T\angle A \cong \mathrm{m}_T\angle O = \pi_T/2$, but $\triangle MOD \not\cong_T \triangle CAB$ because $d_T(C, B) = 2 \neq d_T(M, D)$.

b. There are many examples. One such example is $\triangle PIG$ with vertices $P(-1, 1)$, $I(0, 0)$, and $G(2, 0)$. Then $d_T(P, I) = d_T(M, O) = 2$, $d_T(I, G) = d_T(O, D) = 2$, and $d_T(P, G) = d_T(M, D) = 4$, but $\triangle MOD \not\cong_T \triangle PIG$ because $\mathrm{m}_T\angle PIG > \mathrm{m}_T\angle MOD$.

8. **a**. To use the formula, note that $d_E(O, A) = 3$, $d_E(O, B) = 5$, $d_T(O, A) = 3$, $d_T(O, B) = 7$, $\cos\angle AOB = \frac{3}{5}$, and $\sin\angle AOB = \frac{4}{5}$. Then $\cos_T\angle AOB = \frac{3\cdot5}{3\cdot7}\cdot\frac{3}{5} = \frac{3}{7}$. (Similarly, $\sin_T\angle AOB = \frac{3\cdot5}{3\cdot7}\cdot\frac{4}{5} = \frac{4}{7}$.)

b. Consider the diagram below.



Because $\overrightarrow{OA}$ is parallel to the x-axis, $d_E(O, A) = d_T(O, A)$. Then

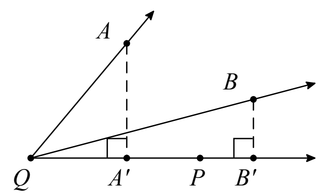
$$\cos_T\angle AOB = \frac{d_E(O,A)\cdot d_E(O,B)}{d_T(O,A)\cdot d_T(O,B)}\cos\angle AOB = \frac{d_E(O,B)}{d_T(O,B)}\cos\angle AOB.$$

Let B' be the foot of the perpendicular from B to $\overleftrightarrow{OA}$, so that $d_T(O, B) = d_E(O, B') + d_E(B', B)$. Then

$$\cos_T\angle AOB = \frac{d_E(O,B)}{d_E(O,B') + d_E(B',B)}\cos\angle AOB = \frac{1}{\frac{d_E(O,B')}{d_E(O,B)} + \frac{d_E(B,B')}{d_E(O,B)}}\cos\angle AOB.$$

Substituting $|\cos\angle AOB|$ for $\frac{d_E(O,B')}{d_E(O,B)}$ and $|\sin\angle AOB|$ for $\frac{d_E(B,B')}{d_E(O,B)}$ yields the desired result. (The reader may wish to show that, under these conditions, $\cos_T\angle AOB+\sin_T\angle AOB=1$.)

c. Let the feet of the altitudes from A and B to $\overleftrightarrow{PQ}$ be A' and B', respectively, as shown in the diagram below.



Then $d_T(Q,A)=d_E(Q,A')+d_E(A',A)$ and $d_T(Q,B)=d_E(Q,B')+d_E(B',B)$. Using the definition yields

$$\begin{aligned}\cos_T\angle AQB &= \frac{d_E(Q,A)\cdot d_E(Q,B)}{d_T(Q,A)\cdot d_T(Q,B)}\cos\angle AQB\\ &= \frac{1}{\left(\dfrac{d_E(Q,A')+d_E(A,A')}{d_E(Q,A)}\right)\cdot\left(\dfrac{d_E(Q,B')+d_E(B,B')}{d_E(Q,B)}\right)}\cos\angle AQB\\ &= \frac{1}{\left(\dfrac{d_E(Q,A')}{d_E(Q,A)}+\dfrac{d_E(A,A')}{d_E(Q,A)}\right)\cdot\left(\dfrac{d_E(Q,B')}{d_E(Q,B)}+\dfrac{d_E(B,B')}{d_E(Q,B)}\right)}\cos\angle AQB.\end{aligned}$$

On the other hand, $\frac{d_E(Q,A')}{d_E(Q,A)}=|\cos\angle AQP|$, $\frac{d_E(A',A)}{d_E(Q,A)}=|\sin\angle AQP|$, and similarly $\frac{d_E(Q,B')}{d_E(Q,B)}=|\cos\angle BQP|$, $\frac{d_E(B',B)}{d_E(Q,B)}=|\sin\angle BQP|$. Hence the right side can be rewritten as

$$\frac{\cos\angle AQB}{(|\cos\angle AQP|+|\sin\angle AQP|)(|\cos\angle BQP|+|\sin\angle BQP|)},$$

which is the desired result.

Alternate solution: Let the feet of the altitudes from A and B to $\overleftrightarrow{PQ}$ be A' and B', respectively. Then

$$\begin{aligned}d_T(Q,A) &= d_E(Q,A')+d_E(A',A)\\ &= d_E(Q,A)\cdot|\cos\angle AQP|+d_E(Q,A)\cdot|\sin\angle AQP|\\ &= d_E(Q,A)\cdot\big(|\cos\angle AQP|+|\sin\angle AQP|\big),\end{aligned}$$

so $d_E(Q,A)/d_T(Q,A)=\big(|\cos\angle AQP|+|\sin\angle AQP|\big)^{-1}$. The analogous formula for $d_E(Q,B)/d_T(Q,B)$ is derived in the same way. Substituting these expressions into the definition of the taxi-cosine yields

$$\begin{aligned}\cos_T\angle AQB &= \frac{d_E(Q,A)\cdot d_E(Q,B)}{d_T(Q,A)\cdot d_T(Q,B)}\cos\angle AQB\\ &= \frac{\cos\angle AQB}{(|\cos\angle AQP|+|\sin\angle AQP|)(|\cos\angle BQP|+|\sin\angle BQP|)},\end{aligned}$$

which is the desired result.

9. For these three problems, the following lemma will be useful:

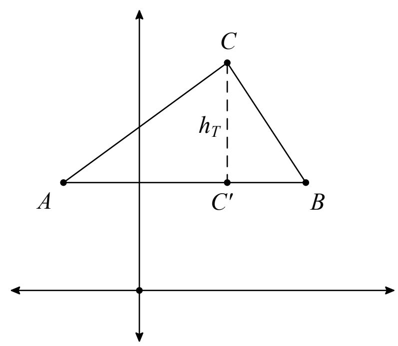
Lemma: Suppose that $\triangle PQR$ is a right triangle with right angle at Q and $\overline{PQ}$ parallel to the x-axis. Then $\cos_T\angle P=\frac{d_T(P,Q)}{d_T(P,R)}$.

Proof of Lemma: Because $\overline{PQ}$ is parallel to the x-axis, the result from 8b applies, and $\cos_T \angle P = \frac{\cos \angle P}{|\cos \angle P| + |\sin \angle P|}$. In this situation, $\cos \angle P = \frac{d_E(P,Q)}{d_E(P,R)}$ and $\sin \angle P = \frac{d_E(Q,R)}{d_E(P,R)}$, so substitute to obtain the following:

$$\begin{aligned}\cos_T \angle P &= \frac{\frac{d_E(P,Q)}{d_E(P,R)}}{\left|\frac{d_E(P,Q)}{d_E(P,R)}\right| + \left|\frac{d_E(Q,R)}{d_E(P,R)}\right|} \\ &= \frac{d_E(P,Q)}{|d_E(P,Q)| + |d_E(Q,R)|}.\end{aligned}$$

However, P and Q lie on the same horizontal line and Q and R lie on the same vertical line, so $d_E(P,Q) = d_T(P,Q)$ and $d_T(P,R) = d_E(P,Q) + d_E(Q,R)$, yielding the desired result. □

Consider the diagram below.



a. Because $d_T(A,C) = d_T(A,C') + d_T(C',C)$, it follows that $d_T(C,C') = d_T(A,C) - d_T(A,C')$ and analogously that $d_T(C,C') = d_T(B,C) - d_T(B,C')$. Then $d_T(B,C) = d_T(A,C) + d_T(A,B) - 2 \cdot d_T(A,C') = b_T + c_T - 2d_T(A,C')$. The lemma applies to angle A in $\triangle AC'C$, and $\cos_T \angle CAC' = \dfrac{d_T(A,C')}{d_T(A,C)}$ implies $d_T(A,C') = d_T(A,C) \cdot \cos_T \angle A$. Substituting this expression for $d_T(A,C')$ into the equation for $d_T(B,C)$ yields the first equation.

b. The second equation is derived similarly. As before, $d_T(A,C) = d_T(B,C) + d_T(A,B) - 2 \cdot d_T(A,C') = a_T + c_T - 2d_T(A,C')$. Again, apply the lemma to angle B in $\triangle BC'C$ to obtain $\cos_T \angle CBC' = \dfrac{d_T(B,C')}{d_T(B,C)}$, yielding $d_T(B,C') = d_T(B,C) \cdot \cos_T \angle B$. Substituting this result into the previous equation yields $d_T(A,C) = a_T + c_T - 2a_T \cos_T \angle B$.

c. Apply the definition of taxi-cosine to obtain $\cos_T \angle ACB = \dfrac{d_E(A,C) \cdot d_E(B,C)}{d_T(A,C) \cdot d_T(B,C)} \cos \angle ACB$. By the Law of Cosines applied to $\triangle ABC$, $\cos \angle C = \dfrac{(d_E(B,C))^2 + (d_E(A,C))^2 - (d_E(A,B))^2}{2 \cdot d_E(A,C) \cdot d_E(B,C)}$. Hence

$$\begin{aligned}\cos_T \angle ACB &= \frac{d_E(A,C) \cdot d_E(B,C)}{d_T(A,C) \cdot d_T(B,C)} \cdot \frac{(d_E(B,C))^2 + (d_E(A,C))^2 - (d_E(A,B))^2}{2 \cdot d_E(A,C) \cdot d_E(B,C)} \\ &= \frac{(d_E(B,C))^2 + (d_E(A,C))^2 - (d_E(A,B))^2}{2 \cdot d_T(A,C) \cdot d_T(B,C)}.\end{aligned}$$

Use the Pythagorean Theorem in $\triangle AC'C$ and $\triangle BC'C$ to obtain

$$\cos_T \angle ACB = \frac{(d_E(B,C'))^2 + (d_E(C,C'))^2 + (d_E(A,C'))^2 + (d_E(C,C'))^2 - (d_E(A,C') + d_E(B,C'))^2}{2 \cdot d_T(A,C) \cdot d_T(B,C)}.$$

By algebra, this equation is equivalent to $\cos_T \angle ACB = \dfrac{h_T^2 - p_T(c_T - p_T)}{a_T \cdot b_T}$, where $p_T = d_T(A,C')$.

Now, obtain expressions for h_T, p_T and $c_T - p_T$ by algebraic manipulation of previously-found equations relating lengths of sides.

First, find an expression for p_T. Note that $d_T(A,C) = b_T = a_T + c_T - 2a_T \cos_T \angle B$, which implies $b_T = a_T + c_T - 2a_T\left(\frac{d_E(B,C)\cdot(c_T - p_T)}{a_T\cdot(c_T-p_T)}\cdot\frac{c_T-p_T}{d_E(B,C)}\right)$. After cancellation, $b_T = a_T + c_T - 2(c_T - p_T) \rightarrow$ $b_T = a_T - c_T + 2p_T \rightarrow p_T = \frac{b_T + c_T - a_T}{2}$.

A similar manipulation to the equation $b_T = a_T + c_T - 2(c_T - p_T)$ yields $c_T - p_T = \frac{a_T + c_T - b_T}{2}$.

Now, notice that $h_T = a_T - (c_T - p_T) = a_T - \frac{a_T + c_T - b_T}{2}$, which simplifies to $\frac{2a_T}{2} - \frac{a_T + c_T - b_T}{2} = \frac{a_T - c_T + b_T}{2}$, as needed. Substitution yields

$$\cos_T \angle ACB = \frac{\left(\frac{b_T + c_T - a_T}{2}\right)^2 - \left(\frac{b_T + c_T - a_T}{2}\right)\left(\frac{a_T + c_T - b_T}{2}\right)}{a_T \cdot b_T},$$

which simplifies to $\cos_T \angle ACB = \frac{a_T^2 + b_T^2 - c_T(a_T + b_T)}{2\cdot a_T \cdot b_T}$.

This equation can be algebraically manipulated to obtain the desired result.

Alternate solution: To simplify the computations, introduce notation for various Euclidean lengths. Let

$$a = d_E(B,C), \qquad b = d_E(A,C), \qquad c = d_E(A,B) = c_T;$$
$$p = d_E(A,C') = d_T(A,C') = b\cos A;$$
$$q = d_E(B,C') = d_T(B,C') = a\cos B.$$

Then

$$a_T = q + h_T, \qquad b_T = p + h_T, \qquad c_T = p + q.$$

a. From the definition of the taxi-cosine, $b_T \cos_T A = b_T \cdot \frac{b\cdot c}{b_T\cdot c_T}\cos A = b\cos A = p$. Therefore $b_T + c_T - 2b_T\cos_T A = (p + h_T) + (p + q) - 2p = q + h_T = a_T$.

b. Similarly, $a_T \cos_T B = q$, so $a_T + c_T - 2a_T\cos_T B = (q + h_T) + (p + q) - 2q = b_T$.

c. The Euclidean radian measures of the angles add up to π, so $\cos C = \cos(\pi - A - B) = -\cos(A+B) = \sin A\,\sin B - \cos A\,\cos B$. Substitute into the definition of the taxi-cosine and multiply by $a_T b_T$:

$$\begin{aligned}
a_T b_T \cos_T C &= a_T b_T \frac{ab}{a_T b_T}(\sin A\,\sin B - \cos A\,\cos B)\\
&= (b\sin A)(a\sin B) - (b\cos A)(a\cos B)\\
&= h_T^2 - pq\\
&= h_T^2 - (b_T - h_T)(a_T - h_T)\\
&= (a_T + b_T)h_T - a_T b_T.
\end{aligned}$$

Therefore $a_T^2 + b_T^2 - 2a_T b_T \cos_T C = a_T^2 + b_T^2 - 2(a_T + b_T)h_T + 2a_T b_T = (a_T + b_T)(a_T + b_T - 2h_T) = (a_T + b_T)c_T$. Solving for c_T gives the desired result.

10. In general, the area of a triangle ABC is equal to half of the area of the parallelogram determined by any two sides of triangle ABC. The definition of taxi-sine, namely that $\sin_T \angle AOB = \frac{d_E(O,A)\cdot d_E(O,B)}{d_T(O,A)\cdot d_T(O,B)}\sin\angle AOB$, implies that $d_T(O,A)\cdot d_T(O,B)\cdot\sin_T\angle AOB = d_E(O,A)\cdot d_E(O,B)\cdot\sin\angle AOB$. The latter expression is equal to $|\overrightarrow{OA}\times\overrightarrow{OB}|_T$. Thus the cross-product of two vectors can be interpreted in a taxicab space as it is in a

Euclidean space!

Now, express the area of the triangle ABC in three different ways:

$$[ABC] = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|_T}{2} = \frac{|\overrightarrow{AB} \times \overrightarrow{BC}|_T}{2} = \frac{|\overrightarrow{AC} \times \overrightarrow{BC}|_T}{2}.$$

Notice also that $|\overrightarrow{AB} \times \overrightarrow{AC}|_T = c_T \cdot b_T \cdot \sin_T A$ and $|\overrightarrow{AB} \times \overrightarrow{BC}|_T = c_T \cdot a_T \cdot \sin_T B$ and $|\overrightarrow{AC} \times \overrightarrow{BC}|_T = b_T \cdot a_T \cdot \sin_T C$. By substitution and multiplication by 2, $c_T \cdot b_T \cdot \sin_T A = c_T \cdot a_T \cdot \sin_T B = b_T \cdot a_T \cdot \sin_T C$. Dividing by the nonzero quantity $a_T b_T c_T$ and taking reciprocals leads to the desired result.

Alternate solution: As in the solution to problem 9, let a, b, and c denote the Euclidean lengths of the sides. With this notation, the definition of the taxi-sine gives

$$\sin_T C = \frac{ab}{a_T b_T} \sin C = \frac{ab \sin C}{a_T b_T}, \quad \text{thus} \quad \frac{c_T}{\sin_T C} = \frac{a_T b_T c_T}{abc} \frac{c}{\sin C}.$$

Hence the Taxicab Law of Sines follows from the Euclidean version, multiplying by $\dfrac{a_T b_T c_T}{abc}$.

Note: Taxicab geometry is an active area of mathematical research, so many important questions remain unanswered. Students who are interested in taxicab geometry might find the following sources helpful; both were inspirational in the creation of this Power Question.

Eugene Krause, *Taxicab Geometry: An Adventure in Non-Euclidean Geometry*, Dover, 1975.

Ayşe Bayar, Süheyla Ekmekçi and Münevver Özcan, *On Trigonometric Functions and Cosine and Sine Rules in Taxicab Plane*, International Electronic Journal of Geometry, **2** (2009), 17-24. Retrieved from http://www.iejgeo.com/matder/dosyalar/makale-11/paper2(bayar-ekmekci-ozcan).pdf.

6 Individual Problems

Problem 1. Given that a, b, and c are positive integers such that $a^b \cdot b^c$ is a multiple of 2016, compute the least possible value of $a + b + c$.

Problem 2. Triangle ABC is isosceles. An ant begins at A, walks exactly halfway along the perimeter of $\triangle ABC$, and then returns directly to A, cutting through the interior of the triangle. The ant's path surrounds exactly 90% of the area of $\triangle ABC$. Compute the maximum possible value of $\tan A$.

Problem 3. Compute $\dfrac{\lfloor \sqrt[4]{1} \rfloor \cdot \lfloor \sqrt[4]{3} \rfloor \cdot \lfloor \sqrt[4]{5} \rfloor \cdots \lfloor \sqrt[4]{2015} \rfloor}{\lfloor \sqrt[4]{2} \rfloor \cdot \lfloor \sqrt[4]{4} \rfloor \cdot \lfloor \sqrt[4]{6} \rfloor \cdots \lfloor \sqrt[4]{2016} \rfloor}$.

Problem 4. Compute the number of permutations $x_1, \ldots, x_6$ of the integers $1, \ldots, 6$ such that $x_{i+1} \leq 2x_i$ for all i, $1 \leq i < 6$.

Problem 5. Compute the least possible non-zero value of $A^2 + B^2 + C^2$ such that A, B, and C are integers satisfying $A \log 16 + B \log 18 + C \log 24 = 0$.

Problem 6. In $\triangle LEO$, point J lies on $\overline{LO}$ such that $\overline{JE} \perp \overline{EO}$, and point S lies on $\overline{LE}$ such that $\overline{JS} \perp \overline{LE}$. Given that $JS = 9, EO = 20$, and $JO + SE = 37$, compute the perimeter of $\triangle LEO$.

Problem 7. Compute the least possible area of a non-degenerate right triangle with sides of length $\sin x$, $\cos x$, and $\tan x$, where x is a real number.

Problem 8. Let $P(x)$ be the polynomial $x^3 + Ax^2 + Bx + C$ for some constants A, B, and C. There exist constants D and E such that for all x, $P(x+1) = x^3 + Dx^2 + 54x + 37$ and $P(x+2) = x^3 + 26x^2 + Ex + 115$. Compute the ordered triple (A, B, C).

Problem 9. An n-sided die has the integers between 1 and n (inclusive) on its faces. All values on the faces of the die are equally likely to be rolled. An 8-sided die, a 12-sided die, and a 20-sided die are rolled. Compute the probability that one of the values rolled is equal to the sum of the other two values rolled.

Problem 10. Compute the largest of the three prime divisors of $13^4 + 16^5 - 172^2$.

7 Answers to Individual Problems

Answer 1. 14

Answer 2. $3\sqrt{11}$

Answer 3. $\frac{5}{16}$

Answer 4. 144

Answer 5. 105

Answer 6. 120

Answer 7. $\frac{\sqrt[4]{2}}{2} - \frac{\sqrt[4]{8}}{4}$

Answer 8. $(20, 11, 5)$

Answer 9. $\frac{23}{240}$

Answer 10. 1321

8 Solutions to Individual Problems

Problem 1. Given that a, b, and c are positive integers such that $a^b \cdot b^c$ is a multiple of 2016, compute the least possible value of $a + b + c$.

Solution 1. Factoring yields $2016 = 2^5 \cdot 3^2 \cdot 7$, which means that $2 \cdot 3 \cdot 7 = 42 \mid ab$. To minimize $a + b$, first look for cases where $ab = 42$. If $a = 2$, then $b = 21$, so setting $c = 2$ is sufficient, yielding 25 for the sum. If $a = 3$, then $b = 14$, but c must be 5 (or greater) to satisfy the condition that $2^5 \mid a^b b^c$, yielding $a + b + c = 22$. If $a = 6$ and $b = 7$, then $c = 1$ is sufficient, yielding $a + b + c = 14$. This value is minimal because $a = 7$ forces $b = 6$ and $c \geq 5$, while the next larger value of a is 14. Thus the minimal value of $a + b + c$ is **14**.

Problem 2. Triangle ABC is isosceles. An ant begins at A, walks exactly halfway along the perimeter of $\triangle ABC$, and then returns directly to A, cutting through the interior of the triangle. The ant's path surrounds exactly 90% of the area of $\triangle ABC$. Compute the maximum possible value of $\tan A$.

Solution 2. Point A cannot be the vertex of the isosceles triangle, because if it were, the ant would reach the midpoint of the base, and then the area surrounded by the ant's path would be half the triangle's area rather than 90%. So A is one of the two base angles. Let P be the point reached by the ant and, without loss of generality, let $m\angle B$ be the triangle's vertex angle. Note that by the triangle inequality, P lies on $\overline{BC}$. There are two cases to consider: either the ant passes through vertex B first, making $[APB]$ greater than $[APC]$, or the ant passes through vertex C first, making $[APC]$ greater than $[APB]$. In the first case, because $\triangle APB$ and $\triangle APC$ have collinear bases (along $\overline{BC}$) and the same altitude (from point A), $\frac{[APB]}{[APC]} = \frac{PB}{PC}$. Because $\frac{[APB]}{[APC]} = \frac{90\%}{10\%} = 9$, $\frac{PB}{PC} = 9$. Without loss of generality, set $PB = 9$ and $PC = 1$. Then $AB = 10$, and from $AB + PB = \frac{1}{2}(AB + BC + AC)$, obtain $AC = 18$. By the Pythagorean Theorem, the altitude to $\overline{AC}$ is $\sqrt{19}$, and $\tan A = \frac{\sqrt{19}}{9} \approx 0.5$. In the second case, analogous reasoning yields $PB = 1$, $PC = 9$, and $AC = 2$, so that $\tan A = \frac{\sqrt{99}}{1} > 0.5$. Thus the answer is $\mathbf{3\sqrt{11}}$.

Problem 3. Compute $\dfrac{\lfloor \sqrt[4]{1} \rfloor \cdot \lfloor \sqrt[4]{3} \rfloor \cdot \lfloor \sqrt[4]{5} \rfloor \cdots \lfloor \sqrt[4]{2015} \rfloor}{\lfloor \sqrt[4]{2} \rfloor \cdot \lfloor \sqrt[4]{4} \rfloor \cdot \lfloor \sqrt[4]{6} \rfloor \cdots \lfloor \sqrt[4]{2016} \rfloor}$.

Solution 3. Notice that $\lfloor \sqrt[4]{n-1} \rfloor = \lfloor \sqrt[4]{n} \rfloor$ except when n is a perfect fourth power. So all of the factors in the product can be ignored except for $\frac{\lfloor \sqrt[4]{15} \rfloor}{\lfloor \sqrt[4]{16} \rfloor}$, $\frac{\lfloor \sqrt[4]{255} \rfloor}{\lfloor \sqrt[4]{256} \rfloor}$, $\frac{\lfloor \sqrt[4]{1295} \rfloor}{\lfloor \sqrt[4]{1296} \rfloor}$. These fractions simplify to $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{6}$, respectively, and their product is $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}$, or $\mathbf{\frac{5}{16}}$.

Problem 4. Compute the number of permutations $x_1, \ldots, x_6$ of the integers $1, \ldots, 6$ such that $x_{i+1} \leq 2x_i$ for all i, $1 \leq i < 6$.

Solution 4. First notice that some pairs are impossible: 1 can only be followed by 2, and 2 can only be followed by 1, 3, or 4. These observations suggest that the problem can be divided into several cases:

Case 1: $x_6 = 1$. If $x_5 = 2$, then there are 4! arrangements of the other numbers. If $x_5 \neq 2$, then holding either 23 or 24 as a consecutive pair yields 4! permutations. So there are $3 \cdot 24 = 72$ possible arrangements.

Case 2: $x_5 = 1$ and $x_6 = 2$. Then there are $4! = 24$ arrangements of the other numbers.

Case 3: Either 123 or 124 occur in that order. Then there are 4! permutations of each cluster and the other three numbers, yielding $2 \cdot 4! = 48$ possible arrangements.

Hence the desired number of permutations of the numbers is **144**.

Problem 5. Compute the least possible non-zero value of $A^2 + B^2 + C^2$ such that A, B, and C are integers satisfying $A \log 16 + B \log 18 + C \log 24 = 0$.

Solution 5. Use the laws of logarithms to obtain $(4 \log 2)A + (\log 2 + 2 \log 3)B + (3 \log 2 + \log 3)C = 0$. Collecting terms with common logarithm coefficients yields $(4A + B + 3C) \log 2 + (2B + C) \log 3 = 0$. Because 3 is not a rational power of 2, in order to satisfy this equation with integral A, B, C, both $4A + B + 3C = 0$ and $2B + C = 0$. The second equation is equivalent to $C = -2B$, yielding $4A + B - 6B = 0$ or $4A = 5B$. Thus A is divisible by 5 and B is divisible by 4. Set $A = 5$, $B = 4$, and $C = -8$ to obtain $5^2 + 4^2 + (-8)^2 = \mathbf{105}$.

Problem 6. In $\triangle LEO$, point J lies on $\overline{LO}$ such that $\overline{JE} \perp \overline{EO}$, and point S lies on $\overline{LE}$ such that $\overline{JS} \perp \overline{LE}$. Given that $JS = 9, EO = 20$, and $JO + SE = 37$, compute the perimeter of $\triangle LEO$.

Solution 6. Optimistically, one might hope that $SE = 12$ and $JO = 25$, and in fact this turns out to be the case! Let $JO = x$ and $SE = y$. Then $JE^2 = 9^2 + y^2 = 81 + y^2$ and $JE^2 + 20^2 = x^2$, hence $y^2 + 481 = x^2$. Thus $x^2 - y^2 = 481$. Because $x + y = 37$, $x - y = 481/37 = 13$. Hence $2x = 50$, $x = 25$, and $y = 12$. That information by itself does not solve the problem, but the fact that the sides of right triangles JES and JOE are known makes a combination of angle-chasing and trigonometry feasible.

Let $\mathrm{m}\angle JOE = \theta$. Then $\mathrm{m}\angle JES = \theta$ because triangles JSE and JEO are both $3-4-5$ right triangles. Then $\mathrm{m}\angle L = 180° - (\mathrm{m}\angle LEO + \mathrm{m}\angle LOE) = 180° - (90° + \theta + \theta) = 90° - 2\theta$. Hence $\tan L = \frac{1}{\tan 2\theta}$ and $\sin L = \cos 2\theta$. Using double-angle identities for tangent and sine yields $\tan L = \frac{1-\tan^2\theta}{2\tan\theta}$ and $\sin L = \cos^2\theta - \sin^2\theta$. In this case, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, and $\tan\theta = \frac{3}{4}$. Substitute these values into the preceding equations to obtain $\tan L = \frac{7/16}{6/4} = \frac{7}{24}$, while $\sin L = \frac{7}{25}$. Then $LS = \frac{9}{\tan L} = \frac{216}{7}$ and $LJ = \frac{9}{\sin L} = \frac{225}{7}$. Thus $LS + LJ = \frac{441}{7} = 63$. With $SE = 12, EO = 20, JO = 25$, the perimeter is $\mathbf{120}$.

Alternate Solution: Let $t = SE$. Then $JE^2 = t^2 + 9^2 = (37 - t)^2 - 20^2$, which implies that $t = 12$. Note that $\triangle JEO \sim \triangle JSE$. Let $\alpha = \mathrm{m}\angle EOJ$; then $\mathrm{m}\angle JES = \alpha$ and $\mathrm{m}\angle LJS = 2\alpha$. Thus $\sin 2\alpha = 2(\sin\alpha)(\cos\alpha) = \frac{24}{25}$, and $\triangle LJS$ is similar to a 7–24–25 triangle. Letting $x = \frac{JS}{9}$, it follows that $LJ + LS = 25x + 24x = 49x = 63$. Thus the perimeter of $\triangle LEO$ is $63 + 37 + 20 = \mathbf{120}$.

Problem 7. Compute the least possible area of a non-degenerate right triangle with sides of length $\sin x$, $\cos x$, and $\tan x$, where x is a real number.

Solution 7. Let $\triangle ABC$ be a right triangle with hypotenuse $\overline{BC}$ and side lengths $\sin x, \cos x$, and $\tan x$. There are three cases to consider, depending on the length chosen for BC. If $BC = \tan x$, then $\tan^2 x = \sin^2 x + \cos^2 x = 1 \Rightarrow \tan x = 1 \Rightarrow x = 45°$. In this case, ABC would be an isosceles right triangle with legs of length $\sin 45° = \frac{\sqrt{2}}{2}$, so $[ABC]$ would be $\frac{AB \cdot AC}{2} = \frac{1}{4}$.

Because $\sin x$, $\cos x$, and $\tan x$ must all be positive,

$$\tan x = \frac{\sin x}{\cos x} \geq \frac{\sin x}{1} = \sin x.$$

Thus it cannot be the case that $BC = \sin x$, because the hypotenuse is the longest side of the triangle.
Finally, if $BC = \cos x$, then $[ABC] = \frac{AB \cdot AC}{2} = \frac{\sin^2 x}{2\cos x}$ and $\cos^2 x = \sin^2 x + \tan^2 x$. The latter equation can be rewritten as

$$\begin{aligned}
\cos^4 x &= (\sin^2 x)(\cos^2)x + \sin^2 x \\
\cos^4 x &= (1 - \cos^2 x)\cos^2 x + (1 - \cos^2 x) \\
2\cos^4 x &= 1,
\end{aligned}$$

and so $\cos x = \frac{1}{\sqrt[4]{2}} = \frac{\sqrt[4]{8}}{2}$, $\sin^2 x = 1 - \frac{\sqrt{2}}{2}$. Thus

$$[ABC] = \frac{\sqrt[4]{2}}{2} - \frac{\sqrt[4]{8}}{4} = \frac{1}{4}\left(2\sqrt[4]{2} - \sqrt[4]{8}\right).$$

The last step is to compare the value above with $\frac{1}{4}$. To see that the value above is less than $\frac{1}{4}$, observe that $4\sqrt[4]{2} < 5$ because $(4\sqrt[4]{2})^4 = 512 < 625 = 5^4$. Next, observe that $2\sqrt[4]{8} > 3$ because $(2\sqrt[4]{8})^4 = 2^7 = 128 > 81 = 3^4$. Put these facts together to see that

$$\frac{1}{4}\left(2\sqrt[4]{2} - \sqrt[4]{8}\right) = \frac{1}{8}\left(4\sqrt[4]{2} - 2\sqrt[4]{8}\right) < \frac{1}{8}(5-3) = \frac{1}{4}.$$

Thus the answer is $\boldsymbol{\frac{\sqrt[4]{2}}{2} - \frac{\sqrt[4]{8}}{4}}$.

Problem 8. Let $P(x)$ be the polynomial x^3+Ax^2+Bx+C for some constants A, B, and C. There exist constants D and E such that for all x, $P(x+1) = x^3 + Dx^2 + 54x + 37$ and $P(x+2) = x^3 + 26x^2 + Ex + 115$. Compute the ordered triple (A, B, C).

Solution 8. Plug $x = 0$ into the given equation involving $P(x+2)$ to obtain $P(2) = 115$. Next, plug in $x = 1$ into the given equation involving $P(x+1)$ to obtain $P(2) = 1 + D + 54 + 37$. Thus $D = 115 - (1 + 54 + 37) = 23$. Then $P(x) = P((x-1)+1) = (x-1)^3 + 23(x-1)^2 + 54(x-1) + 37 = x^3 + 20x^2 + 11x + 5$, hence $(A, B, C) = \mathbf{(20, 11, 5)}$. It can be verified that $E = 103$.

Problem 9. An n-sided die has the integers between 1 and n (inclusive) on its faces. All values on the faces of the die are equally likely to be rolled. An 8-sided die, a 12-sided die, and a 20-sided die are rolled. Compute the probability that one of the values rolled is equal to the sum of the other two values rolled.

Solution 9. Let a, b, and c be the values rolled on the 8-, 12-, and 20-sided dice, respectively. Call a triple (a, b, c) of rolls *great* if one roll equals the sum of the other two.

Imagine that the first two dice are rolled (so that a and b are determined), and then the 20-sided die is rolled. Then the triple (a, b, c) is great if either $c = a + b$, $c = a - b$, or $c = b - a$. The first case is possible for any pair of rolls (a, b), and the probability is $\frac{1}{20}$ for each such pair. The second and third cases are only possible if $a > b$ or if $b > a$, respectively, but again, in each of these cases, the probability of obtaining the proper value of c is $\frac{1}{20}$. So the total probability of rolling a great triple satisfying either $c = a - b$ or $c = b - a$ is simply $\frac{1}{20} \cdot P(a > b \text{ or } b > a)$. But $P(a > b \text{ or } b > a) = 1 - P(a = b) = 1 - \frac{8}{96} = 1 - \frac{1}{12} = \frac{11}{12}$. (This value is simply the probability that whatever number was rolled on the 8-sided die also comes up on the 12-sided die.) So the total probability of rolling a great triple is $\frac{1}{20} + \frac{11}{240} = \mathbf{\frac{23}{240}}$.

Problem 10. Compute the largest of the three prime divisors of $13^4 + 16^5 - 172^2$.

Solution 10. Let $N = 13^4 + 16^5 - 172^2$. Notice that $16^5 = (2^4)^5 = 2^{20}$ and $13^4 - 172^2 = 169^2 - 172^2 = (169 - 172)(169 + 172) = -3 \cdot 341 = -1023 = 1 - 2^{10}$. Thus $N = 2^{20} - 2^{10} + 1$. Multiply by $1025 = 2^{10} + 1$ to obtain

$$\begin{aligned} 1025 \cdot N &= (2^{20} - 2^{10} + 1)(2^{10} + 1) \\ &= 2^{30} + 1. \end{aligned}$$

The right-hand side can be refactored via the identity $4u^4+1=(2u^2+2u+1)(2u^2-2u+1)$, using $u=2^7$:

$$\begin{aligned}
1025\cdot N &= (2\cdot 2^{14}+2\cdot 2^7+1)(2\cdot 2^{14}-2\cdot 2^7+1)\\
5^2\cdot 41\cdot N &= (32768+256+1)(32768-256+1)\\
&= 33025\cdot 32513\\
&= (25\cdot 1321)(41\cdot 793)\\
N &= 1321\cdot 793.
\end{aligned}$$

Searching for small prime divisors leads to $793=13\cdot 61$, so **1321** is the largest prime factor of N.

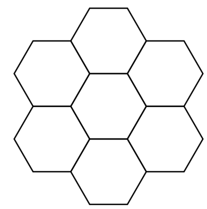
9 Relay Problems

Relay 1-1 Compute the number of lattice points on the graph of $y = (x - 153)^{(153-x^2)}$.

Relay 1-2 Let $T = TNYWR$. Let r and s be the zeros of $x^2 - 4Tx + T^2$. Compute $\sqrt{r} + \sqrt{s}$.

Relay 1-3 Let $T = TNYWR$. Among all triples of integers (a, b, c) satisfying $2^a + 4^b = 8^c$, compute the least value of $a + b + c$ greater than T^2.

Relay 2-1 Compute the number of ways to assign the integers $\{1, 2, \ldots, 7\}$ to each of the hexagons in the figure to the right such that every pair of numbers in edge-adjacent hexagons is relatively prime.



Relay 2-2 Let $T = TNYWR$. In right triangle ABC with hypotenuse $\overline{BC}$, $AB = T$ and $BC = \frac{2T}{\sqrt{3}}$. A circle passes through A and is tangent to $\overline{BC}$ at its midpoint. Compute the radius of the circle.

Relay 2-3 Let $T = TNYWR$. Compute the number of ways to make \$$T$ from an unlimited supply of \$10 bills, \$5 bills, and \$1 bills.

10 Relay Answers

Answer 1-1 27

Answer 1-2 $9\sqrt{2}$

Answer 1-3 169

Answer 2-1 72

Answer 2-2 24

Answer 2-3 9

11 Relay Solutions

Relay 1-1 Compute the number of lattice points on the graph of $y = (x - 153)^{(153-x^2)}$.

Solution 1-1 For an integer x, $(x-153)^{(153-x^2)}$ will be an integer if and only if one of the following three conditions is satisfied:

(1) $153 - x^2 \geq 0$,

(2) $x - 153 = -1$ or 1,

(3) $x - 153 = 0$ and $153 - x^2 > 0$.

For the first case, there are 25 solutions for $-12 \leq x \leq 12$. For the second case, there are two solutions ($x = 152$ or 154). There are no solutions for the third case, as $153 - 153^2 < 0$. Hence there are **27** lattice points on the graph.

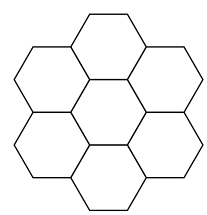
Relay 1-2 Let $T = TNYWR$. Let r and s be the zeros of $x^2 - 4Tx + T^2$. Compute $\sqrt{r} + \sqrt{s}$.

Solution 1-2 Let $u = \sqrt{r} + \sqrt{s}$. Then $u^2 = r + s + 2\sqrt{rs}$. The sum of the zeros of $x^2 - 4Tx + T^2$ is $4T$ and their product is T^2. Thus $u^2 = 4T + 2|T|$. With $T = 27$, $u^2 = 6 \times 27$, so $u = \mathbf{9\sqrt{2}}$.

Relay 1-3 Let $T = TNYWR$. Among all triples of integers (a, b, c) satisfying $2^a + 4^b = 8^c$, compute the least value of $a + b + c$ greater than T^2.

Solution 1-3 Rewrite the equation with common bases to obtain $2^a + 2^{2b} = 2^{3c}$. The only case in which the sum of two integer powers of 2 equals a third integer power of 2 occurs when the summands have equal exponents, so $a = 2b$, $3c = a + 1$, and $a + b + c = a + \frac{a}{2} + \frac{a+1}{3} = \frac{11a+2}{6}$. For all three values to be integers, a must be an even number of the form $3k - 1$. As $T = 9\sqrt{2}$, $T^2 = 162$ and $\frac{11a+2}{6} \geq 162 \Rightarrow 11a + 2 \geq 972 \Rightarrow a \geq \frac{970}{11} > 88$. The least even integer value a of the form $3k - 1$ greater than 88 is 92, and the sum is $92 + 46 + 31 = \mathbf{169}$.

Relay 2-1 Compute the number of ways to assign the integers $\{1, 2, \ldots, 7\}$ to each of the hexagons in the figure to the right such that every pair of numbers in edge-adjacent hexagons is relatively prime.



Solution 2-1 No two even numbers can be adjacent, so the three even numbers must be in alternating hexagons on the boundary. There are two ways to select alternating hexagons on the boundary, and six ways to assign the even numbers to the three hexagons. The number 3 cannot be adjacent to 6, hence the 3 must be opposite the 6. Then the remaining odd numbers can be assigned in any way in the remaining three hexagons. Hence the total number of arrangements is $2 \times 6 \times 6 = \mathbf{72}$.

Relay 2-2 Let $T = TNYWR$. In right triangle ABC with hypotenuse $\overline{BC}$, $AB = T$ and $BC = \frac{2T}{\sqrt{3}}$. A circle passes through A and is tangent to $\overline{BC}$ at its midpoint. Compute the radius of the circle.

Solution 2-2 Note that $\triangle ABC$ is a 30-60-90 right triangle, so $AC = \frac{T}{\sqrt{3}}$. Let point C' be obtained by reflecting point C about $\overline{AB}$ so that $\triangle CBC'$ is equilateral. Note that the circle which passes through A and which is tangent to $\overline{BC}$ at its midpoint happens to be the incircle of $\triangle CBC'$! The radius of the incircle is one-third the altitude of $\triangle CBC'$, so the radius is $\frac{T}{3}$. As $T = 72$, the radius is **24**.

Relay 2-3 Let $T = TNYWR$. Compute the number of ways to make \$$T$ from an unlimited supply of \$10 bills, \$5 bills, and \$1 bills.

Solution 2-3 Let $T = 10t + 5f + n$, where t, f, and n are non-negative integers representing the number of tens, fives, and ones, respectively. Then $2t + f \leq \lfloor \frac{T}{5} \rfloor$, with the remaining total consisting of \$1 bills. The number of ways to make \$$T$ is equal to the number of lattice points (t, f) contained in the triangle bounded by the lines $t = 0$, $f = 0$, and $2t + f = \lfloor \frac{T}{5} \rfloor$. When $t = 0$, there are $\lfloor \frac{T}{5} \rfloor + 1$ possible values of f. Incrementing t by one reduces by two the number of values of f for which there is a solution. As $T = 24$, $\lfloor \frac{T}{5} \rfloor + 1 = 5$, and there are $5 + 3 + 1 = \mathbf{9}$ solutions: $(t, f, n) = (0, 0, 24), (0, 1, 19), (0, 2, 14), (0, 3, 9), (0, 4, 4), (1, 0, 14), (1, 1, 9), (1, 2, 4)$, and $(2, 0, 4)$.

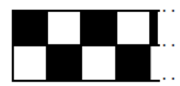
12 Super Relay

1. Given that a, b, and c are positive integers with $ab = 20$ and $bc = 16$, compute the positive difference between the maximum and minimum possible values of $a + b + c$.

2. Let $T = TNYWR$. Mary circles a number in one of the first six rows of Pascal's Triangle, shown at right. Jim then circles another number — possibly having the same value as Mary's number — but Jim's circle cannot coincide with Mary's circle. For example, if Mary circles the top "1", then Jim can circle any number other than the top "1". The product of the two circled numbers is then computed. Compute the number of possible <u>distinct</u> products that are **greater** than T.

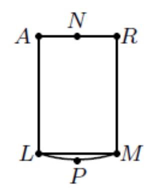
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

3. Let $T = TNYWR$. The diagram at right shows part of a $2 \times T$ checkerboard pattern. A 1×3 tile is randomly placed on the board so that it completely covers three unit squares. Compute the probability that two of the covered squares are black.



4. Let $T = TNYWR$. Leo rides his bike at $16T$ miles per hour and Carla and Rita both ride their bikes at 12 miles per hour. Leo and Carla start at the points $(0, 0)$ and $(60, 0)$, respectively, and they begin biking towards each other (along the x-axis), where the units of both coordinate axes are in miles. At the same time, Rita starts biking from the point (a, b), where $a > 0$, and she is riding along the line $x = a$, towards the x-axis. Compute the value of a such that Leo, Carla, and Rita will meet at the same moment.

5. Let $T = TNYWR$. In rectangle $ARML$, diagrammed at right (not drawn to scale), N is the midpoint of $\overline{AR}$, $NL = 40$, and $AN = T$. Arc $\widehat{LPM}$ is an arc of a circle centered at N, P is the midpoint of arc $\widehat{LPM}$, and arc $\widehat{LP'M}$ is the reflection of $\widehat{LPM}$ across $\overline{LM}$. Given that P' is the midpoint of arc $\widehat{LP'M}$, compute NP'.



6. Let $T = TNYWR$. Let p and q be distinct primes and for each positive integer n, let $d(n)$ be the number of positive divisors of n. Compute the least possible positive integer k such that the quotient $\dfrac{d(p^{T+2}q^k)}{d(p^{T+1}q^2)}$ is an integer.

7. Let $T = TNYWR$. An ARML team of 15 students contains k boys and $15 - k$ girls. The value of k satisfies the equation $\binom{T+1}{3} = 325k$. Compute the probability that a randomly chosen student from the team is a boy.

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15. A dartboard is made up of two concentric circles that have radii 20 and 16. A dart is thrown at random and hits the board. Compute the probability that the dart lands in the circle of radius 16.

14. Let $T = TNYWR$. Let $A = 2^K, R = A^{20}, M = R^{16}$, and $L = M^T$. Given that K and L are positive integers, compute the least possible value of A.

13. Let $T = TNYWR$. Compute $2^{\log_T 8} - 8^{\log_T 2}$.

12. Let $T = TNYWR$. Compute the value of k such that the following system of equations has a solution.

$$\begin{aligned} x + y &= T \\ 20x + 16y &= 8 \\ kx + 20y &= 16. \end{aligned}$$

11. Let $T = TNYWR$. John has T distinct baseball cards and Benson has $2T$ distinct baseball cards, which include the same T cards that John has. John randomly chooses two cards from Benson's deck. The probability that exactly one of the chosen cards is **not** in John's deck can be expressed in the form $\frac{T}{K}$. Compute K.

10. Let $T = TNYWR$. In circle O, perpendicular chords $\overline{AR}$ and $\overline{ML}$ intersect at N. Given that $AN = T$, $RN = 5$, and $MN = 25$, compute the area of circle O.

9. Let $T = TNYWR$, and let $K = \left\lfloor \frac{T}{\pi} \right\rfloor$. Consider the sequence defined by $a_1 = 20$, $a_2 = 16$, and for $n \geq 3$, a_n is the units digit of $a_{n-1} + a_{n-2}$. Compute $\dfrac{a_K}{10}$.

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8. Let t be the number you will receive from position 7 and let s be the number you will receive from position 9. In $\triangle ABC$, point H lies on $\overline{BC}$ such that $\overline{AH} \perp \overline{BC}$. Given that the sides of $\triangle ABC$ are integers, $\tan \angle B = t$, and $\sin \angle CAH = s$, compute the least possible perimeter of $\triangle ABC$.

13 Super Relay Answers

1. 24

2. 6

3. $\frac{1}{2}$

4. 24

5. 24

6. 25

7. $\frac{8}{15}$

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15. $\frac{16}{25}$

14. 32

13. 0

12. 28

11. 55

10. 949π

9. $\frac{3}{5}$

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8. 48

14 Super Relay Solutions

Problem 1. Given that a, b, and c are positive integers with $ab = 20$ and $bc = 16$, compute the positive difference between the maximum and minimum possible values of $a + b + c$.

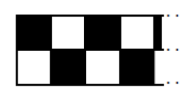
Solution 1. Note that 5 must divide a, hence a must be either $5, 10$, or 20. Thus the possible solutions (a, b, c) are $(5, 4, 4)$, $(10, 2, 8)$, and $(20, 1, 16)$. The first of these gives the minimum possible sum $a + b + c$ of 13 while the last of these gives the maximum possible sum $a + b + c$ of 37. Thus the answer is $37 - 13 = \mathbf{24}$.

Problem 2. Let $T = TNYWR$. Mary circles a number in one of the first six rows of Pascal's Triangle, shown at right. Jim then circles another number — possibly having the same value as Mary's number — but Jim's circle cannot coincide with Mary's circle. For example, if Mary circles the top "1", then Jim can circle any number other than the top "1". The product of the two circled numbers is then computed. Compute the number of possible <u>distinct</u> products that are **greater** than T.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Solution 2. Before receiving a value of T, it makes sense to list out all the possible products. There are **6** products that are greater than T $(= 24)$: 25 (5×5), 30 $(5 \times 6 = 3 \times 10)$, 40 (4×10), 50 (5×10), 60 (6×10), and 100 (10×10).

Problem 3. Let $T = TNYWR$. The diagram at right shows part of a $2 \times T$ checkerboard pattern. A 1×3 tile is randomly placed on the board so that it completely covers three unit squares. Compute the probability that two of the covered squares are black.

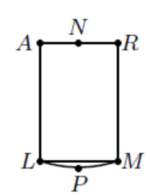


Solution 3. Note that for any possible position of the 1×3 tile's left boundary, there are exactly two placements of the tile (in the first row or in the second row). Exactly one of these placements will cover two white squares and one black square, and the other will cover two black squares and one white square. Because these options are equally likely, the probability that the 1×3 tile covers two black squares is therefore $\mathbf{\frac{1}{2}}$ (independent of the value of T).

Problem 4. Let $T = TNYWR$. Leo rides his bike at $16T$ miles per hour and Carla and Rita both ride their bikes at 12 miles per hour. Leo and Carla start at the points $(0, 0)$ and $(60, 0)$, respectively, and they begin biking towards each other (along the x-axis), where the units of both coordinate axes are in miles. At the same time, Rita starts biking from the point (a, b), where $a > 0$, and she is riding along the line $x = a$, towards the x-axis. Compute the value of a such that Leo, Carla, and Rita will meet at the same moment.

Solution 4. First note that because Carla and Rita ride at the same speed, Carla should ride a distance of b (i.e., Rita's initial y-coordinate) and Leo should ride a distance of $a = 60 - b$ in order for them to meet at the same time. If k is the ratio of Leo's speed to Carla's and Rita's speed, then $\frac{60-b}{b} = \frac{a}{60-a} = r$, hence $a = \frac{60r}{1+r} = \frac{240T}{3+4T}$. With $T = \frac{1}{2}$, it follows that $a = \mathbf{24}$.

Problem 5. Let $T = TNYWR$. In rectangle $ARML$, diagrammed at right (not drawn to scale), N is the midpoint of $\overline{AR}$, $NL = 40$, and $AN = T$. Arc $\widehat{LPM}$ is an arc of a circle centered at N, P is the midpoint of arc $\widehat{LPM}$, and arc $\widehat{LP'M}$ is the reflection of $\widehat{LPM}$ across $\overline{LM}$. Given that P' is the midpoint of arc $\widehat{LP'M}$, compute NP'.



Solution 5. Let $\overline{NP}$ intersect $\overline{LM}$ at Q. Because $NP = NL = 40$, it follows that $PQ = 40 - AL$. Because $P'Q = PQ$, it follows that $NP' = AL - (40 - AL) = 2 \cdot AL - 40$. With $T = 24$, it follows that $\triangle NAL$ is similar to a $3-4-5$ triangle, hence $AL = 32$, and $NP' = \mathbf{24}$.

Problem 6. Let $T = TNYWR$. Let p and q be distinct primes and for each positive integer n, let $d(n)$ be the number of positive divisors of n. Compute the least possible positive integer k such that the quotient $\dfrac{d(p^{T+2}q^k)}{d(p^{T+1}q^2)}$ is an integer.

Solution 6. Note that $d(p^{T+2}q^k) = (T+3)(k+1)$ because a positive divisor of $p^{T+2}q^k$ is of the form $p^x q^y$, where x and y are integers satisfying $0 \leq x \leq T+2$ and $0 \leq y \leq k$. Similarly, $d(p^{T+1}q^2) = 3(T+2)$. Thus the quotient is $\dfrac{(T+3)(k+1)}{3(T+2)}$. Note that $T+3$ and $T+2$ are relatively prime, so either $k = T+1$ if 3 divides $T+3$ or $k = 3T+5$ if 3 does not divide $T+3$. With $T = 24$, the former case applies, thus $k = \mathbf{25}$.

Problem 7. Let $T = TNYWR$. An ARML team of 15 students contains k boys and $15-k$ girls. The value of k satisfies the equation $\binom{T+1}{3} = 325k$. Compute the probability that a randomly chosen student from the team is a boy.

Solution 7. With $T = 25$, $\binom{26}{3} = \frac{26 \cdot 25 \cdot 24}{3!} = 2600$. Thus $k = 2600/325 = 8$, and the desired probability is $\frac{8}{15}$.

Problem 15. A dartboard is made up of two concentric circles that have radii 20 and 16. A dart is thrown at random and hits the board. Compute the probability that the dart lands in the circle of radius 16.

Solution 15. The inner circle has radius 16 and therefore has area $\pi \cdot 16^2$. The outer circle has radius 20 and therefore has area $\pi \cdot 20^2$. The probability that a dart lands in the inner circle is the ratio of these areas, so the desired probability is $\frac{\pi \cdot 16^2}{\pi \cdot 20^2} = \left(\frac{16}{20}\right)^2 = \left(\frac{4}{5}\right)^2 = \mathbf{\frac{16}{25}}$.

Problem 14. Let $T = TNYWR$. Let $A = 2^K, R = A^{20}, M = R^{16}$, and $L = M^T$. Given that K and L are positive integers, compute the least possible value of A.

Solution 14. Note that $R = (A^K)^{20} = A^{20K}$, $M = R^{16} = A^{320K}$, and $L = M^T = A^{320KT}$. If T is a positive integer, then $K = 1$ suffices. On the other hand, if T is a rational number (say $\frac{c}{d}$ in lowest terms), but not a positive integer, then $K = \frac{d}{\gcd(320,d)}$. With $T = \frac{16}{25}$, $K = \frac{25}{\gcd(320,25)} = 5$. Thus $A = 2^5 = \mathbf{32}$.

Problem 13. Let $T = TNYWR$. Compute $2^{\log_T 8} - 8^{\log_T 2}$.

Solution 13. Note that $2^{\log_T 8} = 2^{\log_T 2^3} = 2^{3 \log_T 2} = 8^{\log_T 2}$. Thus, so long as T is positive and $T \neq 1$ (to ensure that the function $\log_T$ is defined), the expression $2^{\log_T 8} - 8^{\log_T 2}$ equals 0. Because $T = 32$ satisfies these constraints, the answer is $\mathbf{0}$.

Problem 12. Let $T = TNYWR$. Compute the value of k such that the following system of equations has a solution.

$$\begin{aligned} x + y &= T \\ 20x + 16y &= 8 \\ kx + 20y &= 16. \end{aligned}$$

Solution 12. Subtract sixteen times the first equation from the second equation to obtain $4x = 8 - 16T$. Thus $x = 2 - 4T$. Plug this into the first equation to get $y = T - x = 5T - 2$. Plug in these values for x and y into the third equation to obtain $k(2 - 4T) + 20(5T - 2) = 16 \to k = \dfrac{28 - 50T}{1 - 2T}$. With $T = 0$, it follows that $k = \frac{28}{1} = \mathbf{28}$.

Problem 11. Let $T = TNYWR$. John has T distinct baseball cards and Benson has $2T$ distinct baseball cards, which include the same T cards that John has. John randomly chooses two cards from Benson's deck. The probability that exactly one of the chosen cards is **not** in John's deck can be expressed in the form $\dfrac{T}{K}$. Compute K.

Solution 11. There are $\binom{2T}{2} = T(2T - 1)$ possible selections of two cards that John can choose from Benson's deck. Of those, there will be T^2 pairs of cards, exactly one of which is not in John's deck. Hence the desired probability is $\frac{T^2}{T(2T-1)} = \frac{T}{2T-1}$, hence $K = 2T - 1$. With $T = 28$, $K = \mathbf{55}$.

Problem 10. Let $T = TNYWR$. In circle O, perpendicular chords $\overline{AR}$ and $\overline{ML}$ intersect at N. Given that $AN = T$, $RN = 5$, and $MN = 25$, compute the area of circle O.

Solution 10. Note that by Power of a Point, $LN = \frac{AN \cdot RN}{MN} = \frac{T}{5}$. With $T = 55$, $LN = 11$. Let P be the foot of the perpendicular from O to $\overline{AR}$ and let Q be the foot of the perpendicular from O to $\overline{ML}$. Note that $AP = \frac{1}{2}(55 + 5) = 30$ and $MQ = \frac{1}{2}(25 + 11) = 18$. Because chords $\overline{AR}$ and $\overline{ML}$ are perpendicular, it follows that $OP = QN = MN - MQ = 7$ and that $OQ = PN = PR - NR = 30 - 5 = 25$. Letting r be the radius of circle O, it follows that $r^2 = 30^2 + 7^2 = 18^2 + 25^2 = 949$, hence the answer is $\mathbf{949\pi}$.

Problem 9. Let $T = TNYWR$, and let $K = \left\lfloor \frac{T}{\pi} \right\rfloor$. Consider the sequence defined by $a_1 = 20$, $a_2 = 16$, and for $n \geq 3$, a_n is the units digit of $a_{n-1} + a_{n-2}$. Compute $\dfrac{a_K}{10}$.

Solution 9. Ignoring the tens digits of the first two terms, the sequence is

$$0, 6, 6, 2, 8, 0, 8, 8, 6, 4, 0, 4, 4, 8, 2, 0, 2, 2, 4, 6, 0, 6, 6, 2, \ldots.$$

Note that this sequence is periodic with period 20. Thus when m is an integer, $a_{20m} = 6$. With $T = 949\pi$, $K = 949$, and $a_{940} = 6$. Counting 9 terms into the next block of 20 terms, conclude that $a_{949} = 6$, hence the desired ratio is $\mathbf{\frac{3}{5}}$.

Problem 8. Let t be the number you will receive from position 7 and let s be the number you will receive from position 9. In $\triangle ABC$, point H lies on $\overline{BC}$ such that $\overline{AH} \perp \overline{BC}$. Given that the sides of $\triangle ABC$ are integers, $\tan \angle B = t$, and $\sin \angle CAH = s$, compute the least possible perimeter of $\triangle ABC$.

Solution 8. Let $BH = x$ and $AC = y$. Then $AH = tx$ and $CH = sy$. By the Pythagorean Theorem, $AB = x\sqrt{1 + t^2}$ and $t^2x^2 + s^2y^2 = y^2$, hence $AC = y = \frac{tx}{\sqrt{1-s^2}}$. With $t = \frac{8}{15}$ and $s = \frac{3}{5}$, note that triangles AHB and CHA are similar to 8–15–17 and 3–4–5 triangles, respectively. Setting $x = 15$ gives $AH = 8$, $AB = 17$, $CH = 6$, and $AC = 10$. Thus the minimum perimeter of $\triangle ABC$ is $17 + 10 + (15 + 6) = \mathbf{48}$.

15 Tiebreaker Problems

Problem 1. Compute the least value of N such that there are exactly 43 ordered quadruples of positive integers (a, b, c, d) satisfying $N = a^2 + b^2 + c^2 + d^2$.

Problem 2. Regular octagon $HEPTAGON$ has legs of length 8. Segments $\overline{ET}$ and $\overline{PA}$ intersect at X. Compute the area of heptagon $HEXAGON$.

Problem 3. Compute the base-10 value of $111_2 + 222_3 + 333_4 + \cdots + 888_9$.

16 Tiebreaker Answers

Answer 1. 100

Answer 2. $112 + 112\sqrt{2}$

Answer 3. 2016

17 Tiebreaker Solutions

Problem 1. Compute the least value of N such that there are exactly 43 ordered quadruples of positive integers (a, b, c, d) satisfying $N = a^2 + b^2 + c^2 + d^2$.

Solution 1. To understand how to count the number of ordered quadruples that satisfy the given equation, it is useful to consider a particular value of N and a particular solution to the resulting equation. So, for example, with $N = 18$, one solution is $(1, 2, 2, 3)$. Because of the symmetry of the expression $a^2 + b^2 + c^2 + d^2$, any permutation of $(1, 2, 2, 3)$ is also a solution, so there are $\frac{4!}{2!} = 12$ different quadruples corresponding to the same values; call the set of all such quadruples a *solution class*. This observation suggests starting by identifying different possible sizes of solution classes. Assuming that the initial values satisfy $a \le b \le c \le d$, the different solution classes and their sizes are given in the following table.

Solution class	Size
$a < b < c < d$	$4! = 24$
$a = b < c < d$ $a < b = c < d$ $a < b < c = d$	$\frac{4!}{2!} = 12$
$a = b < c = d$	$\frac{4!}{2!2!} = 6$
$a = b = c < d$ $a < b = c = d$	$\frac{4!}{3!} = 4$
$a = b = c = d$	1

The goal is then to identify the least value of N such that the sizes of the different solution classes sum to 43. Note that 43 is odd, while all but one of the class sizes are even. So for some positive integer m, $N = m^2 + m^2 + m^2 + m^2 = 4m^2 = (2m)^2$. Thus N is not only even, it is the square of an even number. Furthermore, note that all other solution classes except the case $a = b < c = d$ have sizes divisible by four. Because 42 is not divisible by four, there must also be a solution of the form $N = a^2 + a^2 + c^2 + c^2 = 2a^2 + 2c^2$, which implies that half of N is a sum of squares of distinct integers.

These two insights rule out several small cases, because for $m = 1, 2, 3, 4$, $N/2 = 2, 8, 18, 32$ respectively, and none of these is the sum of two *distinct* squares. On the other hand, $m = 5$ yields $N = 100$, and $N/2 = 50 = 1^2 + 7^2$, which is promising! In fact,

$$\begin{aligned}100 &= 9^2 + 3^2 + 3^2 + 1^2\\ &= 8^2 + 4^2 + 4^2 + 2^2\\ &= 7^2 + 5^2 + 5^2 + 1^2,\end{aligned}$$

yielding a total of $3 \cdot 12 + 6 + 1 = 43$ solutions. For $N = 100$, note that if the resulting equation had a solution of the form $a = b = c < d$, then because 100 is not a multiple of 3, d cannot be a multiple of 3. Similarly, if a solution of the form $a < b = c = d$ existed, then a cannot be a multiple of 3. But because none of $(100 - 1^2)/3$, $(100-2^2)/3$, $(100-4^2)/3$, $(100-7^2)/3$, or $(100-8^2)/3$ is a perfect square, and because $(100-5^2)/3 = 25 = 5^2$, corresponds to the case $100 = 5^2 + 5^2 + 5^2 + 5^2$, which has already been accounted for, it follows that there are no solutions of the form $a = b = c < d$ or $a < b = c = d$.

Lastly, for $N = 100$, the equation has no solutions of the form $a < b < c < d$. To see this, note that a perfect square is either congruent to 1 or 0 modulo 4, depending on its parity. Given that $a^2 + b^2 + c^2 + d^2 = 100 \equiv 0 \pmod 4$, it follows that a, b, c and d must all have the same parity. If all are even, then the least possible value of $a^2 + b^2 + c^2 + d^2$ with $a < b < c < d$ would be $2^2 + 4^2 + 6^2 + 8^2 = 120$ which is larger than 100. If all are odd, then the least value is $1^2 + 3^2 + 5^2 + 7^2 = 84 \ne 100$. The second smallest value is $1^2 + 3^2 + 5^2 + 9^2 = 116$. Thus there are no solutions in the solution class $a < b < c < d$, and so the least value of N for which the equation $a^2 + b^2 + c^2 + d^2 = N$ has 43 solutions is **100**.

Problem 2. Regular octagon $HEPTAGON$ has legs of length 8. Segments $\overline{ET}$ and $\overline{PA}$ intersect at X. Compute the area of heptagon $HEXAGON$.

Solution 2. Dissect heptagon $HEXAGON$ into hexagon $HEAGON$ and $\triangle AXE$. First compute the area of hexagon $HEAGON$. Extend $\overline{HE}$ and $\overline{ON}$ to meet at Q, and extend $\overline{ON}$ and $\overline{AG}$ to meet at R. Note that $\triangle HNQ$ and $\triangle GOR$ are isosceles right triangles each with hypotenuse of length 8. Thus each triangle has legs of length $4\sqrt{2}$ and therefore $[HNQ] = [GOR] = \frac{1}{2}(4\sqrt{2})^2 = 16$. Therefore

$$\begin{aligned}
[HEAGON] &= [AEQR] - [HNQ] - [GOR] \\
&= QR \cdot QE - 16 - 16 \\
&= (QN + NO + OR)(QH + HE) - 32 \\
&= (8 + 8\sqrt{2})(8 + 4\sqrt{2}) - 32 \\
&= 96 + 96\sqrt{2}.
\end{aligned}$$

Next, compute $[AXE]$. Draw in altitude $\overline{XY}$ from X to $\overline{EA}$, so that

$$[AXE] = \tfrac{1}{2}EA \cdot XY = (4 + 4\sqrt{2}) \cdot XY.$$

To complete this computation, determine XY as follows. Draw $\overline{HA}$. Because $\overline{HA} \parallel \overline{ET}$ and $\overline{HE} \parallel \overline{XY}$, triangles HEA and XYE are similar. Specifically, $EY = \frac{1}{2}EA$, so $XY = \frac{1}{2}HE = 4$. Thus

$$[AXE] = (4 + 4\sqrt{2}) \cdot 4 = (16 + 16\sqrt{2}).$$

Finally, combine these computations to obtain

$$\begin{aligned}
[HEXAGON] &= [HEAGON] + [AXE] \\
&= (96 + 96\sqrt{2}) + (16 + 16\sqrt{2}) \\
&= \mathbf{112 + 112\sqrt{2}}.
\end{aligned}$$

Problem 3. Compute the base-10 value of $111_2 + 222_3 + 333_4 + \cdots + 888_9$.

Solution 3. Note that $\underline{m}\,\underline{m}\,\underline{m}_{m+1} = (m+1)^3 - 1$. So the desired sum S is

$$\begin{aligned}
S &= (2^3 - 1) + \cdots + (9^3 - 1) \\
&= (2^3 + \cdots + 9^3) - 8 \\
&= (1^3 + 2^3 + \cdots + 9^3) - 8 - 1^3.
\end{aligned}$$

Use the identity $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = \left(\frac{n(n+1)}{2}\right)^2$ to obtain

$$S = \left(\frac{9 \cdot 10}{2}\right)^2 - 9 = 45^2 - 9 = \mathbf{2016}.$$