ARML Competition 2019

George Reuter, Head Writer
Chris Jeuell, Lead Editor
Evan Chen
Paul Dreyer
Edward Early
Zuming Feng
Zachary Franco
Silas Johnson
Winston Luo
Jason Mutford
Andy Niedermaier
Graham Rosby
Eric Wepsic

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Sponsored By:

D E Shaw & Co

1 Team Problems

Problem 1. The points $(1, 2, 3)$ and $(3, 3, 2)$ are vertices of a cube. Compute the product of all possible distinct volumes of the cube.

Problem 2. Eight students attend a Harper Valley ARML practice. At the end of the practice, they decide to take selfies to celebrate the event. Each selfie will have either two or three students in the picture. Compute the minimum number of selfies so that each pair of the eight students appears in exactly one selfie.

Problem 3. Compute the least positive value of t such that

$$\operatorname{Arcsin}(\sin(t)),\ \operatorname{Arccos}(\cos(t)),\ \operatorname{Arctan}(\tan(t))$$

form (in some order) a three-term arithmetic progression with a nonzero common difference.

Problem 4. In non-right triangle ABC, distinct points P, Q, R, and S lie on $\overline{BC}$ in that order such that $\angle BAP \cong \angle PAQ \cong \angle QAR \cong \angle RAS \cong \angle SAC$. Given that the angles of $\triangle ABC$ are congruent to the angles of $\triangle APQ$ in some order of correspondence, compute $\mathrm{m}\angle B$ in degrees.

Problem 5. Consider the system of equations

$$\begin{aligned}\log_4 x + \log_8(yz) &= 2\\ \log_4 y + \log_8(xz) &= 4\\ \log_4 z + \log_8(xy) &= 5.\end{aligned}$$

Given that xyz can be expressed in the form 2^k, compute k.

Problem 6. A complex number z is selected uniformly at random such that $|z| = 1$. Compute the probability that z and z^{2019} both lie in Quadrant II in the complex plane.

Problem 7. Compute the least positive integer n such that the sum of the digits of n is five times the sum of the digits of $(n + 2019)$.

Problem 8. Compute the greatest real number K for which the graphs of

$$(|x| - 5)^2 + (|y| - 5)^2 = K \quad \text{and} \quad (x - 1)^2 + (y + 1)^2 = 37$$

have exactly two intersection points.

Problem 9. To *morph* a sequence means to replace two terms a and b with $a+1$ and $b-1$ if and only if $a+1 < b-1$, and such an operation is referred to as a morph. Compute the least number of morphs needed to transform the sequence $1^2, 2^2, 3^2, \ldots, 10^2$ into an arithmetic progression.

Problem 10. Triangle ABC is inscribed in circle ω. The tangents to ω at B and C meet at point T. The tangent to ω at A intersects the perpendicular bisector of $\overline{AT}$ at point P. Given that $AB = 14$, $AC = 30$, and $BC = 40$, compute $[PBC]$.

2 Answers to Team Problems

Answer 1. 216

Answer 2. 12

Answer 3. $\dfrac{3\pi}{4}$

Answer 4. $\dfrac{45}{2}$ (or $22\frac{1}{2}$ or 22.5)

Answer 5. $\dfrac{66}{7}$ (or $9\frac{3}{7}$)

Answer 6. $\dfrac{505}{8076}$

Answer 7. 7986

Answer 8. 29

Answer 9. 56

Answer 10. $\dfrac{800}{3}$ (or $266\frac{2}{3}$ or $266.\overline{6}$)

3 Solutions to Team Problems

Problem 1. The points $(1,2,3)$ and $(3,3,2)$ are vertices of a cube. Compute the product of all possible distinct volumes of the cube.

Solution 1. The distance between points $A(1,2,3)$ and $B(3,3,2)$ is $AB = \sqrt{(3-1)^2 + (3-2)^2 + (2-3)^2} = \sqrt{6}$. Denote by s the side length of the cube. Consider three possibilities.

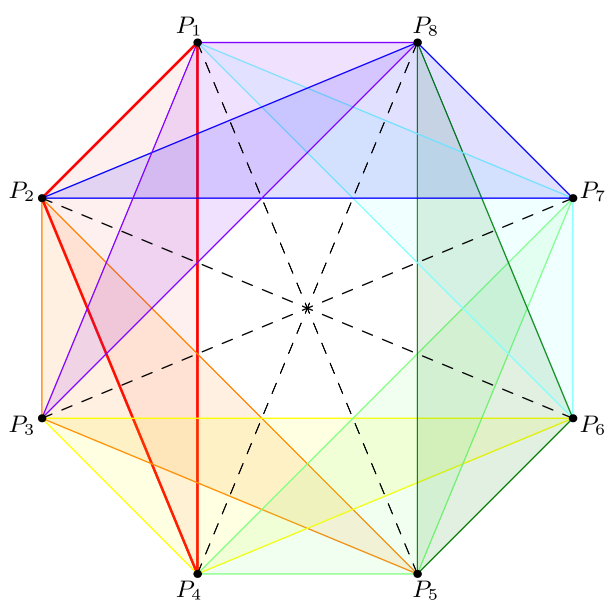
- If $\overline{AB}$ is an edge of the cube, then $AB = s$, so one possibility is $s_1 = \sqrt{6}$.
- If $\overline{AB}$ is a face diagonal of the cube, then $AB = s\sqrt{2}$, so another possibility is $s_2 = \sqrt{3}$.
- If $\overline{AB}$ is a space diagonal of the cube, then $AB = s\sqrt{3}$, so the last possibility is $s_3 = \sqrt{2}$.

The answer is then $s_1^3 s_2^3 s_3^3 = (s_1 s_2 s_3)^3 = 6^3 = \mathbf{216}$.

Problem 2. Eight students attend a Harper Valley ARML practice. At the end of the practice, they decide to take selfies to celebrate the event. Each selfie will have either two or three students in the picture. Compute the minimum number of selfies so that each pair of the eight students appears in exactly one selfie.

Solution 2. The answer is 12. To give an example in which 12 selfies is possible, consider regular octagon $P_1P_2P_3P_4P_5P_6P_7P_8$. Each vertex of the octagon represents a student and each of the diagonals and sides of the octagon represents a pair of students. Construct eight triangles $P_1P_2P_4$, $P_2P_3P_5$, $P_3P_4P_6$, $\ldots$, $P_8P_1P_3$. Each of the segments in the forms of $\overline{P_iP_{i+1}}$, $\overline{P_iP_{i+2}}$, $\overline{P_iP_{i+3}}$ appears exactly once in these eight triangles. Taking 8 three-person selfies (namely $\{P_1, P_2, P_4\}$, $\{P_2, P_3, P_5\}$, $\ldots$, $\{P_8, P_1, P_3\}$) and 4 two-person selfies (namely $\{P_1, P_5\}$, $\{P_2, P_6\}$, $\{P_3, P_7\}$, $\{P_4, P_8\}$) gives a total of 12 selfies, completing the desired task.

A diagram of this construction is shown below. Each of the eight triangles is a different color, and each of the two-person selfies is represented by a dotted diameter.



It remains to show fewer than 12 selfies is impossible. Assume that the students took x three-person selfies and y two-person selfies. Each three-person selfie counts 3 pairs of student appearances (in a selfie), and each two-person selfie counts 1 pair of student appearances (in a selfie). Together, these selfies count $3x + y$ pairs of student appearances. There are $\binom{8}{2} = 28$ pairs of student appearances. Hence $3x + y = 28$. The number of selfies is $x + y = 28 - 2x$, so it is enough to show that $x \leq 8$.

Assume for contradiction there are $x \geq 9$ three-person selfies; then there are at least $3 \cdot 9 = 27$ (individual) student appearances on these selfies. Because there are 8 students, some student s_1 had at least $\lceil 27/8 \rceil$ appearances; that is, s_1 appeared in at least 4 of these three-person selfies. There are $2 \cdot 4 = 8$ (individual) student appearances other than s_1 on these 4 selfies. Because there are only 7 students besides s_1, some other student s_2 had at least $\lceil 8/7 \rceil$ (individual) appearances on these 4 selfies; that is, s_2 appeared (with s_1) in at least 2 of these 4 three-person selfies, violating the condition that each pair of the students appears in exactly one selfie. Thus the answer is **12**.

Problem 3. Compute the least positive value of t such that

$$\operatorname{Arcsin}(\sin(t)),\ \operatorname{Arccos}(\cos(t)),\ \operatorname{Arctan}(\tan(t))$$

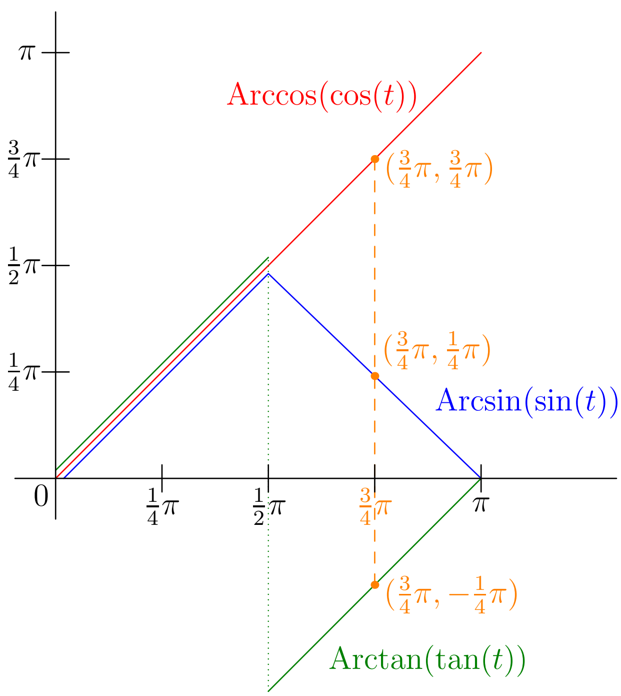
form (in some order) a three-term arithmetic progression with a nonzero common difference.

Solution 3. For $0 \leq t < \pi/2$, all three values are t, so the desired t does not lie in this interval.

For $\pi/2 < t < \pi$,

$$\begin{aligned} \operatorname{Arcsin}(\sin(t)) &= \pi - t \in (0, \pi/2) \\ \operatorname{Arccos}(\cos(t)) &= t \qquad \in (\pi/2, \pi) \\ \operatorname{Arctan}(\tan(t)) &= t - \pi \in (-\pi/2, 0)\,. \end{aligned}$$

A graph of all three functions is shown below.



Thus if the three numbers are to form an arithmetic progression, they should satisfy

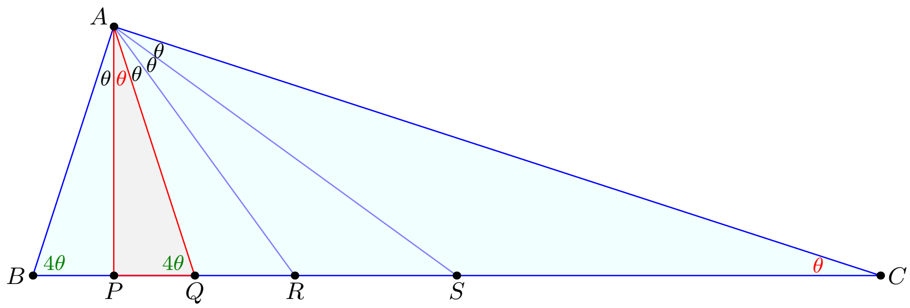
$$t - \pi < \pi - t < t.$$

The three numbers will be in arithmetic progression if and only if $t + (t - \pi) = 2(\pi - t)$, which implies $t = \boldsymbol{\frac{3\pi}{4}}$. Note that if $t = \frac{3\pi}{4}$, the arithmetic progression is $-\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$, as required.

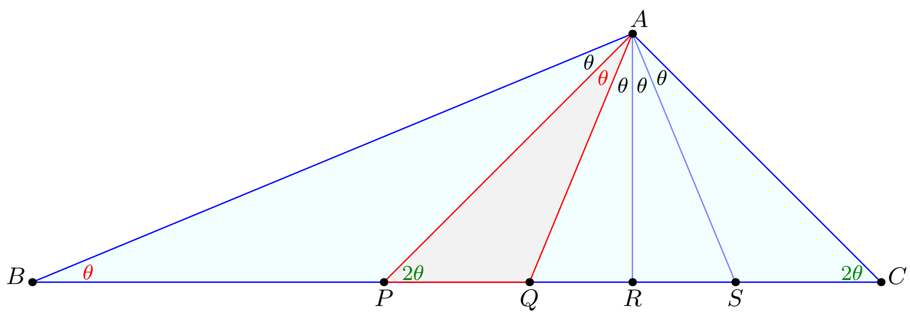
Problem 4. In non-right triangle ABC, distinct points P, Q, R, and S lie on $\overline{BC}$ in that order such that $\angle BAP \cong \angle PAQ \cong \angle QAR \cong \angle RAS \cong \angle SAC$. Given that the angles of $\triangle ABC$ are congruent to the angles of $\triangle APQ$ in some order of correspondence, compute $\mathrm{m}\angle B$ in degrees.

Solution 4. Let $\theta = \frac{1}{5}\mathrm{m}\angle A$. Because $\mathrm{m}\angle PAQ = \theta < 5\theta = \mathrm{m}\angle A$, it follows that either $\mathrm{m}\angle B = \theta$ or $\mathrm{m}\angle C = \theta$. Thus there are two cases to consider.

If $\mathrm{m}\angle C = \theta$, then it follows that $\mathrm{m}\angle AQP = \mathrm{m}\angle QAC + \mathrm{m}\angle ACB = 4\theta$, and hence $\mathrm{m}\angle B = 4\theta$. So $\triangle ABC$ has angles of measures 5θ, 4θ, θ, and thus $\theta = 18°$. However, this implies $\mathrm{m}\angle A = 5\theta = 90°$, which is not the case.



If instead $\mathrm{m}\angle B = \theta$, then it follows that $\mathrm{m}\angle APQ = \mathrm{m}\angle BAP + \mathrm{m}\angle ABP = 2\theta$, and hence $\mathrm{m}\angle C = 2\theta$. So $\triangle ABC$ has angles of measures 5θ, 2θ, θ, and thus $\theta = 22.5°$. Hence $\mathrm{m}\angle B = \theta = \mathbf{22.5°}$.



Problem 5. Consider the system of equations

$$\begin{aligned}\log_4 x + \log_8(yz) &= 2\\ \log_4 y + \log_8(xz) &= 4\\ \log_4 z + \log_8(xy) &= 5.\end{aligned}$$

Given that xyz can be expressed in the form 2^k, compute k.

Solution 5. Note that for $n > 0$, $\log_4 n = \log_{64} n^3$ and $\log_8 n = \log_{64} n^2$. Adding together the three given equations and using both the preceding facts and properties of logarithms yields

$$\begin{aligned}&\log_4(xyz) + \log_8(x^2y^2z^2) = 11\\ \Longrightarrow\ &\log_{64}(xyz)^3 + \log_{64}(xyz)^4 = 11\\ \Longrightarrow\ &\log_{64}(xyz)^7 = 11\\ \Longrightarrow\ &7\log_{64}(xyz) = 11.\end{aligned}$$

The last equation is equivalent to $xyz = 64^{11/7} = 2^{66/7}$, hence the desired value of k is $\mathbf{\dfrac{66}{7}}$.

Problem 6. A complex number z is selected uniformly at random such that $|z| = 1$. Compute the probability that z and z^{2019} both lie in Quadrant II in the complex plane.

Solution 6. For convenience, let $\alpha = \pi/4038$. Denote by

$$0 \le \theta < 2\pi = 8076\alpha$$

the complex argument of z, selected uniformly at random from the interval $[0, 2\pi)$. Then z itself lies in Quadrant II if and only if

$$2019\alpha = \frac{\pi}{2} < \theta < \pi = 4038\alpha.$$

On the other hand, z^{2019} has argument 2019θ, and hence it lies in Quadrant II if and only if there is some integer k with

$$\begin{aligned} &\frac{\pi}{2} + 2k\pi < 2019\theta < \pi + 2k\pi \\ \iff &(4k+1) \cdot \frac{\pi}{2} < 2019\theta < (4k+2) \cdot \frac{\pi}{2} \\ \iff &(4k+1)\alpha < \theta < (4k+2)\alpha. \end{aligned}$$

Because it is also true that $2019\alpha < \theta < 4038\alpha$, the set of θ that satisfies the conditions of the problem is the union of intervals:

$$(2021\alpha, 2022\alpha) \cup (2025\alpha, 2026\alpha) \cup \cdots \cup (4037\alpha, 4038\alpha).$$

There are 505 such intervals, the j^{th} interval consisting of $(4j + 2017)\alpha < \theta < (4j + 2018)\alpha$. Each interval has length α, so the sum of the intervals has length 505α. Thus the final answer is

$$\frac{505\alpha}{2\pi} = \frac{505}{2 \cdot 4038} = \mathbf{\frac{505}{8076}}.$$

Problem 7. Compute the least positive integer n such that the sum of the digits of n is five times the sum of the digits of $(n + 2019)$.

Solution 7. Let $S(n)$ denote the sum of the digits of n, so that solving the problem is equivalent to solving $S(n) = 5S(n + 2019)$. Using the fact that $S(n) \equiv n \pmod 9$ for all n, it follows that

$$\begin{aligned} n &\equiv 5(n + 2019) \equiv 5(n + 3) \pmod 9 \\ 4n &\equiv -15 \pmod 9 \\ n &\equiv 3 \pmod 9. \end{aligned}$$

Then $S(n + 2019) \equiv 6 \pmod 9$. In particular, $S(n + 2019) \ge 6$ and $S(n) \ge 5 \cdot 6 = 30$. The latter inequality implies $n \ge 3999$, which then gives $n + 2019 \ge 6018$. Thus if $n + 2019$ were a four-digit number, then $S(n + 2019) \ge 7$. Moreover, $S(n + 2019)$ can only be 7, because otherwise, $S(n) = 5S(n + 2019) \ge 40$, which is impossible (if n has four digits, then $S(n)$ can be no greater than 36). So if $n + 2019$ were a four-digit number, then $S(n + 2019) = 7$ and $S(n) = 35$. But this would imply that the digits of n are 8, 9, 9, 9 in some order, contradicting the assumption that $n + 2019$ is a four-digit number. On the other hand, if $n + 2019$ were a five-digit number such that $S(n + 2019) \ge 6$, then the least such value of $n + 2019$ is 10005, and indeed, this works because it corresponds to $n = \mathbf{7986}$, the least possible value of n.

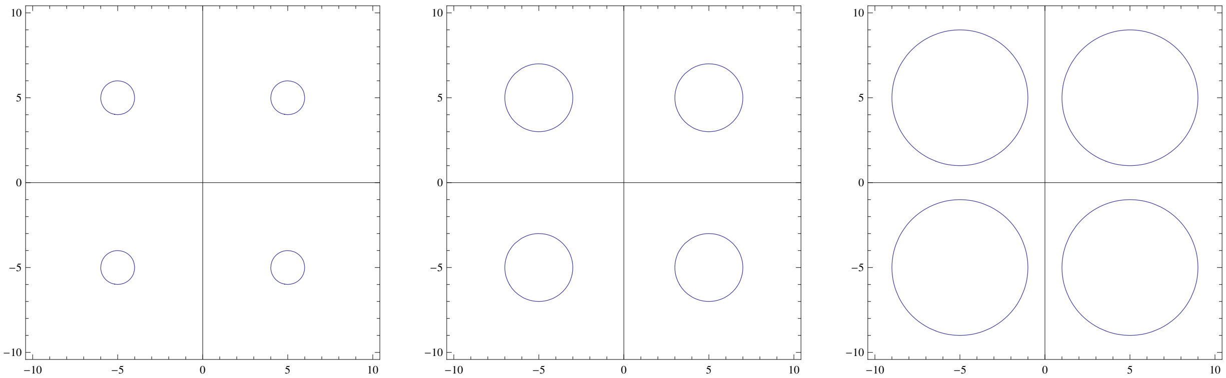
Problem 8. Compute the greatest real number K for which the graphs of

$$(|x| - 5)^2 + (|y| - 5)^2 = K \quad \text{and} \quad (x - 1)^2 + (y + 1)^2 = 37$$

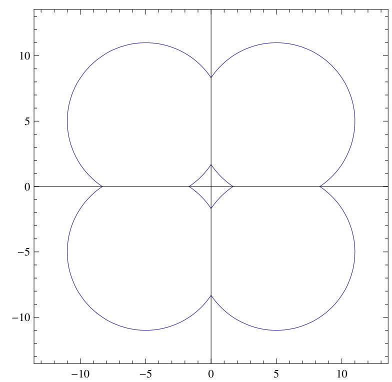
have exactly two intersection points.

Solution 8. The graph of the second equation is simply the circle of radius $\sqrt{37}$ centered at $(1, -1)$. The first graph is more interesting, and its behavior depends on K.

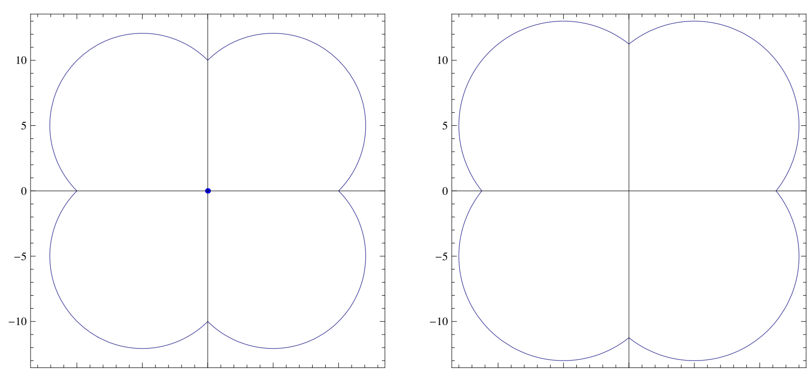
- For small values of K, the first equation determines a set of four circles of radius $\sqrt{K}$ with centers at $(5,5), (5,-5), (-5,5)$, and $(-5,-5)$. Shown below are versions with $K = 1$, $K = 4$, and $K = 16$.



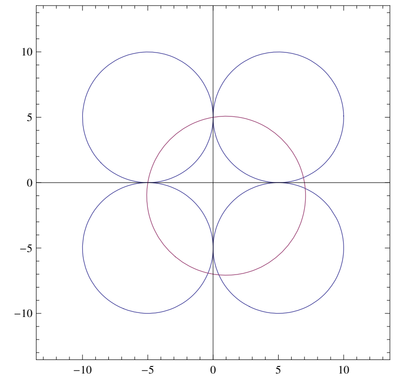
- However, when $K > 25$, the graph no longer consists of four circles! As an example, for $K = 36$, the value $x = 5$ gives $(|y| - 5)^2 = 36$; hence $|y| = -1$ or $|y| = 6$. The first option is impossible; the graph ends up "losing" the portions of the upper-right circle that would cross the x- or y-axes compared to the graph for $(x-5)^2 + (y-5)^2 = 36$. The graph for $K = 36$ is shown below.



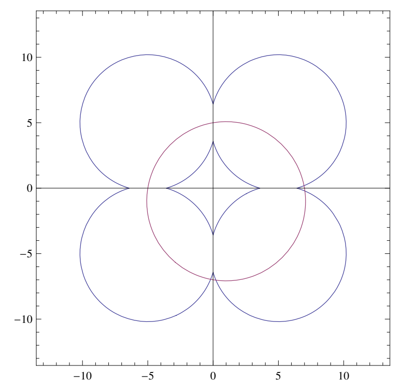
- As K continues to increase, the "interior" part of the curve continues to shrink, until at $K = 50$, it simply comprises the origin, and for $K > 50$, it does not exist. As examples, the graphs with $K = 50$ and $K = 64$ are shown below.



Overlay the graph of the circle of radius $\sqrt{37}$ centered at $(1, -1)$ with the given graphs. When $K = 25$, this looks like the following graph.



Note that the two graphs intersect at $(0, 5)$ and $(-5, 0)$, as well as four more points (two points near the positive x-axis and two points near the negative y-axis). When K is slightly greater than 25, this drops to four intersection points. The graph for $K = 27$ is shown below.



Thus for the greatest K for which there are exactly two intersection points, those two intersection points should be along the positive x- and negative y-axes. If the intersection point on the positive x-axis is at $(h, 0)$, then $(h-1)^2 + (0+1)^2 = 37$ and $(h-5)^2 + (0-5)^2 = K$. Thus $h = 7$ and $K = \mathbf{29}$.

Problem 9. To *morph* a sequence means to replace two terms a and b with $a+1$ and $b-1$ if and only if $a+1 < b-1$, and such an operation is referred to as a morph. Compute the least number of morphs needed to transform the sequence $1^2, 2^2, 3^2, \ldots, 10^2$ into an arithmetic progression.

Solution 9. Call the original sequence of ten squares $T = (1^2, 2^2, \ldots, 10^2)$. A *morphed sequence* is one that can be obtained by morphing T a finite number of times.

This solution is divided into three steps. In the first step, a characterization of the possible final morphed sequences is given. In the second step, a lower bound on the number of steps is given, and in the third step, it is shown that this bound can be achieved.

Step 1. Note the following.

- The sum of the elements of T is $1^2 + 2^2 + \cdots + 10^2 = 385$, and morphs are sum-preserving. So any morphed sequence has sum 385 and a mean of 38.5.
- The sequence T has positive integer terms, and morphs preserve this property. Thus any morphed sequence has positive integer terms.
- The sequence T is strictly increasing, and morphs preserve this property. Thus any morphed sequence is strictly increasing.

Now if the morphed sequence is an arithmetic progression, it follows from the above three observations that it must have the form

$$(38.5 - 4.5d,\ 38.5 - 3.5d,\ \ldots,\ 38.5 + 4.5d)$$

where d is an odd positive integer satisfying $38.5 - 4.5d > 0$. Therefore the only possible values of d are 7, 5, 3, 1; thus there are at most four possibilities for the morphed sequence, shown in the table below. Denote these four sequences by A, B, C, D.

	T	1	4	9	16	25	36	49	64	81	100
$d = 7:$	A	7	14	21	28	35	42	49	56	63	70
$d = 5:$	B	16	21	26	31	36	41	46	51	56	61
$d = 3:$	C	25	28	31	34	37	40	43	46	49	52
$d = 1:$	D	34	35	36	37	38	39	40	41	42	43

Step 2. Given any two sequences $X = (x_1, \ldots, x_{10})$ and $Y = (y_1, \ldots, y_{10})$ with $\sum_{i=1}^{10} x_i = \sum_{i=1}^{10} y_i = 385$, define the *taxicab distance*

$$\rho(X, Y) = \sum_{i=1}^{10} |x_i - y_i|.$$

Observe that if X' is a morph of X, then $\rho(X', Y) \geq \rho(X, Y) - 2$. Therefore the number of morphs required to transform T into some sequence Z is at least $\frac{1}{2}\rho(T, Z)$. Now

$$\frac{1}{2}\rho(T, A) = \frac{1}{2}\sum_{i=1}^{10} \left|i^2 - 7i\right| = 56$$

and also $\rho(T, A) < \min\left(\rho(T, B), \rho(T, C), \rho(T, D)\right)$. Thus at least 56 morphs are needed to obtain sequence A (and more morphs would be required to obtain any of sequences B, C, or D).

Step 3. To conclude, it remains to verify that one can make 56 morphs and arrive from T to A. One of many possible constructions is given below.

T	1	4	9	16	25	36	49	64	81	100
6 morphs	1	4	9	16	25	42	49	58	81	100
2 morphs	1	4	9	16	27	42	49	56	81	100
8 morphs	1	4	9	16	35	42	49	56	73	100
10 morphs	1	4	9	26	35	42	49	56	63	100
2 morphs	1	4	9	28	35	42	49	56	63	98
12 morphs	1	4	21	28	35	42	49	56	63	86
10 morphs	1	14	21	28	35	42	49	56	63	76
6 morphs	7	14	21	28	35	42	49	56	63	70

Therefore the least number of morphs needed to transform T into an arithmetic progression is **56**.

Remark: For step 3, one may prove more generally that any sequence of 56 morphs works as long as both of the following conditions hold:

- each morph increases one of the first six elements and decreases one of the last three elements, and
- at all times, the i^{th} term is at most $7i$ for $i \leq 6$, and at least $7i$ for $i \geq 8$.

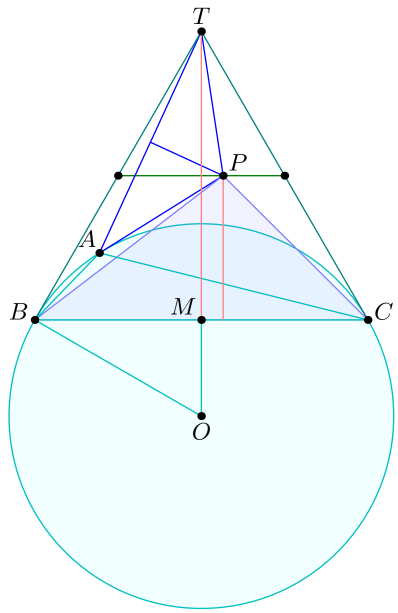
Problem 10. Triangle ABC is inscribed in circle ω. The tangents to ω at B and C meet at point T. The tangent to ω at A intersects the perpendicular bisector of $\overline{AT}$ at point P. Given that $AB = 14$, $AC = 30$, and $BC = 40$, compute $[PBC]$.

Solution 10. To begin, denote by R the radius of ω. The semiperimeter of triangle ABC is 42, and then applying Heron's formula yields

$$[ABC] = \frac{14 \cdot 30 \cdot 40}{4R} = \sqrt{42 \cdot 28 \cdot 12 \cdot 2} = 168$$

from which it follows that $R = \dfrac{14 \cdot 30 \cdot 40}{4 \cdot 168} = 25$.

Now consider the point circle with radius zero centered at T in tandem with the circle ω. Because $PA = PT$, it follows that P lies on the radical axis of these circles. Moreover, the midpoints of $\overline{TB}$ and $\overline{TC}$ lie on this radical axis as well. Thus P lies on the midline of $\triangle TBC$ that is parallel to $\overline{BC}$.



To finish, let O denote the center of ω and M the midpoint of $\overline{BC}$. By considering right triangle TBO with altitude $\overline{BM}$, it follows that $MT \cdot MO = MB^2$, but also $MO = \sqrt{OB^2 - MB^2} = \sqrt{25^2 - 20^2} = 15$, so

$$MT = \frac{MB^2}{MO} = \frac{400}{15} = \frac{80}{3}.$$

Thus the distance from P to $\overline{BC}$ is $\frac{1}{2}MT = \frac{40}{3}$. Finally,

$$[PBC] = \frac{1}{2} \cdot \frac{40}{3} \cdot BC = \mathbf{\frac{800}{3}}.$$

4 Power Question 2019: Elizabeth's Escape

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

Elizabeth is in an "escape room" puzzle. She is in a room with one door which is locked at the start of the puzzle. The room contains n light switches, each of which is initially off. Each minute, she must flip exactly k different light switches (to "flip" a switch means to turn it on if it is currently off, and off if it is currently on). At the end of each minute, if all of the switches are on, then the door unlocks and Elizabeth escapes from the room.

Let $E(n, k)$ be the minimum number of minutes required for Elizabeth to escape, for positive integers n, k with $k \leq n$. For example, $E(2, 1) = 2$ because Elizabeth cannot escape in one minute (there are two switches and one must be flipped every minute) but she can escape in two minutes (by flipping Switch 1 in the first minute and Switch 2 in the second minute). Define $E(n, k) = \infty$ if the puzzle is impossible to solve (that is, if it is impossible to have all switches on at the end of any minute).

For convenience, assume the n light switches are numbered 1 through n.

1. Compute the following.

- **a.** $E(6, 1)$ [1 pt]
- **b.** $E(6, 2)$ [1 pt]
- **c.** $E(7, 3)$ [1 pt]
- **d.** $E(9, 5)$ [1 pt]

2. Find the following in terms of n.

- **a.** $E(n, 2)$ for positive even integers n [2 pts]
- **b.** $E(n, 3)$ for values of n of the form $n = 3a + 2$ where a is a positive integer [2 pts]
- **c.** $E(n, n - 2)$ for $n \geq 5$ [2 pts]

3. Find an integer value of k with $1 < k < 2019$ such that $E(2019, k) = \infty$. [3 pts]

4. **a.** Show that if $n + k$ is even and $\frac{n}{2} < k < n$, then $E(n, k) = 3$. [3 pts]

b. Show that if n is even and k is odd, then $E(n, k) = E(n, n - k)$. [3 pts]

5. Find the following.

- **a.** $E(2020, 1993)$ [3 pts]
- **b.** $E(2001, 501)$ [3 pts]

6. **a.** Show that if n and k are both even and $k \leq \frac{n}{2}$, then $E(n, k) = \lceil \frac{n}{k} \rceil$. [3 pts]

b. Prove that if k is odd and $k \leq \frac{n}{2}$, then either $E(n, k) = \lceil \frac{n}{k} \rceil$ or $E(n, k) = \lceil \frac{n}{k} \rceil + 1$. [4 pts]

7. Find all ordered pairs (n, k) for which $E(n, k) = 3$. [3 pts]

One might guess that in most cases, $E(n,k) \approx \frac{n}{k}$. In light of this guess, define the *inefficiency* of the ordered pair (n,k), denoted $I(n,k)$, as

$$I(n,k) = E(n,k) - \frac{n}{k}$$

if $E(n,k) \neq \infty$. If $E(n,k) = \infty$, then by convention, $I(n,k)$ is undefined.

8. **a.** Compute $I(6,3)$. [1 pt]

b. Compute $I(5,3)$. [1 pt]

c. Find positive integers n and k for which $I(n,k) = \frac{15}{8}$. [2 pts]

d. Prove that for any integer $x > 2$, there exists an ordered pair (n,k) for which $I(n,k) > x$. [3 pts]

9. Let S be the set of values of $I(n,k)$ for all n,k for which $k < \frac{n}{2}$ and $I(n,k)$ is defined. Find the least upper bound of S. Prove that your answer is correct. [4 pts]

10. Find two distinct non-integral positive rational numbers that are not the inefficiency of any ordered pair. That is, find positive rational numbers q_1 and q_2 with $q_1 \neq q_2$ such that neither q_1 nor q_2 is an integer and such that neither q_1 nor q_2 is $I(n,k)$ for any integers n and k. Prove that your answers are correct. [4 pts]

5 Solutions to Power Question

First, notice that a light switch is on if it has been flipped an odd number of times, and off if it has been flipped an even number of times.

We use the notation $\{a_1, a_2, \ldots, a_k\}$ to denote the set of k switches flipped in any given minute.

1. **a.** $E(6,1) = 6$. Note that at least six minutes are required because exactly one switch is flipped each minute. By flipping all six switches (in any order) in the first six minutes, the door will open in six minutes.

b. $E(6,2) = 3$. The sequence $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ will allow Elizabeth to escape the room in three minutes. It is not possible to escape the room in fewer than three minutes because every switch must be flipped, and that requires at least $\frac{6}{2} = 3$ minutes.

c. $E(7,3) = 3$. First, note that $E(7,3) \geq 3$, because after only two minutes, it is impossible to flip each switch at least once. It is possible to escape in three minutes with the sequence $\{1,2,3\}$, $\{1,4,5\}$, and $\{1,6,7\}$.

d. $E(9,5) = 3$. Notice that $E(9,5) \neq 1$ because each switch must be flipped at least once, and only five switches can be flipped in one minute. Notice also that $E(9,5) \neq 2$ because after two minutes, there have been 10 flips, but in order to escape the room, each switch must be flipped at least once, and this requires 9 of the 10 flips. However, the tenth flip of a switch returns one of the nine switches to the off position, so it is not possible for Elizabeth to escape in two minutes. In three minutes, however, Elizabeth can escape with the sequence $\{1,2,3,4,5\}$, $\{1,2,3,6,7\}$, $\{1,2,3,8,9\}$.

2. **a.** If n is even, then $E(n,2) = \frac{n}{2}$. This is the minimum number of minutes required to flip each switch at least once, and Elizabeth can clearly escape in $\frac{n}{2}$ minutes by flipping each switch *exactly* once.

b. If $n = 3a + 2$ $(a \geq 1)$, then $E(n,3) = a + 2$. The minimum number of minutes required to flip each switch once is $a + 1$, but as in Problem 1d, this leaves exactly one "extra flip", so some switch must be flipped exactly twice. However, in $a + 2 = \frac{n+4}{3}$ minutes, Elizabeth can escape by starting with the sequence $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, and flipping each remaining switch exactly once.

c. If $n \geq 5$, then $E(n, n-2) = 3$. Note that Elizabeth cannot flip every switch in one minute, and after two minutes, some switch (in fact, many switches) must be flipped exactly twice. However, Elizabeth can escape in three minutes using the sequence $\{1,4,5,\ldots,n\}$, $\{2,4,5,\ldots,n\}$, $\{3,4,5,\ldots,n\}$.

3. The answer is that k can be any even number between 2 and 2018 inclusive. If Elizabeth escapes, each switch must have been flipped an odd number of times. Because there are 2019 switches, the total number of flips must be odd. However, if an even number of flips are performed each minute, then the total number of flips cannot be odd. Therefore the puzzle is impossible, and $E(2019, k) = \infty$.

Alternate Solution: As in Problem 2a, consider the number of lights that are on after each minute. Flipping an even number of switches either leaves this number unchanged, or increases or decreases it by an even number. Because the puzzle starts with 0 lights on, and 2019 is odd, it is impossible to have 2019 lights on at the end of any minute. Thus $E(2019, k) = \infty$.

4. **a.** First, because $k < n$, not all switches can be flipped in one minute, and so $E(n,k) \neq 1$. Second, because $k > \frac{n}{2}$, some switches must be flipped twice in the first two minutes, and so $E(n,k) \neq 2$.

Here is a strategy by which Elizabeth can escape in three minutes. Note that because $n + k$ is even, it follows that $3k - n = (k+n) + 2(k-n)$ is also even. Then let $3k - n = 2b$. Note that $b > 0$ (because $k > \frac{n}{2}$), and $b < k$ (because $k < n$).

Elizabeth's strategy is to flip switches 1 through b three times each, and the remaining $n - b$ switches once each. This is possible because $n - b = (3k - 2b) - b = 3(k - b)$ is divisible by 3, and $b + \frac{n-b}{3} = b + (k - b) = k$. Therefore, each minute, Elizabeth can flip switches 1 through b, and one-third of the $3(k-b)$ switches from $b + 1$ through n. This allows her to escape in three minutes, as desired.

b. Because n is even, and because each switch must be flipped an odd number of times in order to escape, the total number of flips is even. Because k must be odd, $E(n,k)$ must be even. To show this, consider the case where $E(n,k)$ is odd. If $E(n,k)$ is odd, then an odd number of flips happen an odd number of times, resulting in an odd number of total flips. This is a contradiction because n is even.

Call a switch "non-flipped" in any given minute if it is not among the switches flipped in that minute. Because $E(n,k)$ (i.e., the total number of minutes) is even, and each switch is flipped an odd number of times, each switch must also be non-flipped an odd number of times. Therefore any sequence of flips that solves the "(n,k) puzzle" can be made into a sequence of flips that solves the "$(n,n-k)$" puzzle by interchanging flips and non-flips. These sequences last for the same number of minutes, and therefore $E(n,k)=E(n,n-k)$.

5. **a.** $E(2020,1993)=76$. By the result of Problem 4, conclude that $E(2020,1993)=E(2020,27)$. Compute the latter instead. Because $\frac{2020}{27}>74$, it will require at least 75 minutes to flip each switch once. Furthermore, $E(2020,27)\geq 76$ because the solution to Problem 4 implies that $E(2020,27)$ is even.

To solve the puzzle in exactly 76 minutes, use the following strategy. For the first 33 minutes, flip switch 1, along with the first 26 switches that have not yet been flipped. The end result is that lights 1 through $26\cdot 33+1=859$ are on, and the remaining 1161 lights are off. Note that $1161=27\cdot 43$, so it takes 43 minutes to flip each remaining switch exactly once, for a total of 76 minutes, as desired.

b. $E(2001,501)=5$. First, note that three minutes is not enough time to flip each switch once. In four minutes, Elizabeth can flip each switch once, but has three flips left over. Because there are an odd number of leftover flips to distribute among the 2001 switches, some switch must get an odd number of leftover flips, and thus an even number of total flips. Thus $E(2001,501)>4$.

To solve the puzzle in five minutes, Elizabeth can flip the following sets of switches:

- in the first minute, $\{1,2,3,\ldots,501\}$;
- in the second minute, $\{1,2,3,\ldots,102\}$ and $\{502,503,504,\ldots,900\}$;
- in the third minute, $\{1,2,3,\ldots,102\}$ and $\{901,902,903,\ldots,1299\}$;
- in the fourth minute, $\{1,2,3,\ldots,100\}$ and $\{1300,1301,1302,\ldots,1700\}$;
- in the fifth minute, $\{1,2,3,\ldots,100\}$ and $\{1701,1702,1703,\ldots,2001\}$.

This results in switches $1,2,3,\ldots,100$ being flipped five times, switches 101 and 102 being flipped three times, and the remaining switches being flipped exactly once, so that all the lights are on at the end of the fifth minute.

6. **a.** First, if k divides n, then $E(n,k)=\frac{n}{k}=\lceil\frac{n}{k}\rceil$. Assume then that k does not divide n. Then let $r=\lceil\frac{n}{k}\rceil$, which implies $(r-1)k<n<rk$.

Because $(r-1)k<n$, it follows that $r-1$ minutes are not enough to flip each switch once, so $E(n,k)\geq r$.

Because n and k are even, it follows that rk and $rk-n$ are also even. Then $rk-n=2b$ for some integer $b\geq 1$, and note that $b<k$ because $rk-k<n$. Then the following strategy turns all of the lights on in r minutes.

- In each of the first three minutes, flip switches 1 through b, along with the next $k-b$ switches that have not yet been flipped. This leaves $b+3(k-b)$ lights on, and the rest off. Note that $b+3(k-b)=3k-2b=3k-(rk-n)=n-k(r-3)$.
- In each of the remaining $r-3$ minutes, flip the first k switches that have never been flipped before.

This strategy turns on the remaining $k(r-3)$ lights, and Elizabeth escapes the room after r minutes.

b. First, if k divides n, then $E(n,k)=\frac{n}{k}=\lceil\frac{n}{k}\rceil$. Assume then that k does not divide n. Then let $r=\lceil\frac{n}{k}\rceil$, which implies $(r-1)k<n<rk$.

Because $(r-1)k<n$, it follows that $r-1$ minutes are not enough to flip each switch once, so $E(n,k)\geq r$.

Exactly one of $rk - n$ and $(r+1)k - n$ is even. There are two cases.

Case 1: Suppose $rk - n = 2b$ for some integer $b \geq 1$. As in the solution to Problem 6a, $b < k$. Then the following strategy turns all of the lights on in r minutes.

- In each of the first three minutes, flip switches 1 through b, along with the next $k - b$ switches that have not yet been flipped. This leaves $b + 3(k - b)$ lights on, and the rest off. Note that $b + 3(k - b) = 3k - 2b = 3k - (rk - n) = n - k(r - 3)$.
- In each of the remaining $r - 3$ minutes, flip the first k switches that have never been flipped before.

This strategy turns on the remaining $k(r - 3)$ lights, and Elizabeth escapes the room after r minutes.

Case 2: Suppose $(r+1)k - n = 2b$ for some integer $b \geq 1$. As in the solution to Problem 6a, $b < k$. Then the following strategy turns all of the lights on in $r + 1$ minutes.

- In each of the first three minutes, flip switches 1 through b, along with the next $k - b$ switches that have not yet been flipped. This leaves $b + 3(k - b)$ lights on, and the rest off. Note that $b + 3(k - b) = 3k - 2b = 3k - (rk + k - n) = n - k(r - 2)$.
- In each of the remaining $r - 2$ minutes, flip the first k switches that have never been flipped before.

This strategy turns on the remaining $k(r - 2)$ lights, and Elizabeth escapes the room after $r + 1$ minutes.

7. Consider the parity of n and of k. If n is odd and k is even, then as shown in Problem 3, $E(n,k) = \infty \neq 3$. If n is even and k is odd, then as noted in the solution to Problem 5, $E(n,k)$ is even, so $E(n,k) \neq 3$. If n and k are either both even or both odd and $\frac{n}{3} \leq k < n$, then the argument in Problem 4 shows that $E(n,k) = 3$.

If $k < \frac{n}{3}$, then three minutes is not enough time to flip each switch at least once, so $E(n,k) > 3$.

If $k = n$, then $E(n,k) = 1$.

Therefore $E(n,k) = 3$ if and only if $n + k$ is even (that is, n and k have the same parity) and $\frac{n}{3} \leq k < n$.

8. **a.** $I(6,3) = 0$. By definition, $I(6,3) = E(6,3) - \frac{6}{3}$. Because $3 \mid 6$, $E(6,3) = \frac{6}{3} = 2$, and so $I(6,3) = 2 - 2 = 0$.

b. $I(5,3) = \frac{4}{3}$. By definition, $I(5,3) = E(5,3) - \frac{5}{3}$. By Problem 2b, $E(5,3) = E(3 \cdot 1 + 2, 3) = 1 + 2 = 3$, and so $I(5,3) = 3 - \frac{5}{3} = \frac{4}{3}$.

c. One such pair is $(n,k) = (18,16)$. If $I(n,k) = \frac{15}{8}$, then $E(n,k) - \frac{n}{k} = \frac{15}{8}$. Note that $E(n,k) > 2$ because $k < n$.
Suppose $E(n,k) = 3$. Then $\frac{n}{k} = \frac{9}{8}$. From Problem 7, $E(n,k) = 3$ if and only if n and k have the same parity and $\frac{n}{3} \leq k < n$, so let $n = 18$ and $k = 16$. Then, $I(18,16) = E(18,16) - \frac{18}{16} = 3 - \frac{9}{8} = \frac{15}{8}$, as desired.

d. Let $n = 2x$ and $k = 2x - 1$ for some positive integer x. Then n is even and k is odd, so Problem 4 applies, and $E(n,k) = E(n, n-k) = E(n,1) = 2x$. Therefore $I(n,k) = 2x - \frac{2x}{2x-1}$. Because $x > 2$, it follows that $\frac{2x}{2x-1} < 2 < x$, so $I(n,k) > x$, as desired.

9. The least upper bound of S is 2. First, note that Problems 6a and 6b together show that if $k < \frac{n}{2}$, then $E(n,k)$ is either $\lceil \frac{n}{k} \rceil$ or $\lceil \frac{n}{k} \rceil + 1$ (or ∞ in the case where k is even and n is odd). Therefore $I(n,k) \leq 2$ because $E(n,k)$ is at most $\lceil \frac{n}{k} \rceil + 1$ and $\lceil \frac{n}{k} \rceil + 1 - \frac{n}{k} \leq 2$.

Now it will be shown that $I(n,k)$ can be arbitrarily close to 2. To do this, use values of n and k that are both odd, such that $\lceil \frac{n}{k} \rceil$ is even. Specifically, let $n = 3k + 2$ for k odd; then $\lceil \frac{n}{k} \rceil = 4$. Therefore $E(n,k) = 5$ by Problem 6b, and

$$I(n,k) = 5 - \frac{3k+2}{k} = 2 - \frac{2}{k}.$$

Letting k be a large odd integer, this gives an ordered pair (n,k) with $I(n,k)$ arbitrarily close to 2, as desired.

Combining these two claims shows that the least upper bound of S is 2.

10. It should be noted that integer answers are not possible because no positive integer can be the inefficiency of an ordered pair (n, k). This is because if $I(n, k) = E(n, k) - \frac{n}{k}$ is an integer, then $\frac{n}{k}$ is an integer so $E(n, k) = \frac{n}{k}$ and $I(n, k) = 0$.

To find non-integral rational numbers that cannot be the inefficiency of any ordered pair, start with a simple lemma.

Lemma: Given positive integers n, k, and r, $E(rn, rk) \leq E(n, k)$.

Proof: If n is odd and k is even, then $E(n, k) = \infty$ and the lemma is trivially true. Otherwise there is a sequence of $E(n, k)$ sets of switches to flip which will result in escape. Partition the rn switches into r groups of n switches, and apply the strategy within each of the r groups, which uses rk flips per minute. This will achieve escape in $E(n, k)$ minutes, so $E(rn, rk)$ is at most $E(n, k)$. $\square$

The values $E(n, k)$ and $E(rk, nk)$ can certainly equal one other, as when n is a multiple of k. The inequality can also be strict. For example, $E(4, 3) = E(4, 1) = 4$ but $E(8, 6) = 3$ are obtained from values from earlier work.

One implication of the above lemma is that $I(n, k) \geq I(rn, rk)$.

Now all rational numbers q with $0 < q < 1$ are the inefficiency of some ordered pair (n, k). To show this, let $r = 3 - q = \frac{a}{b}$ in lowest terms. Then $r > 2$ so $a > 2b$, and by Problem 6a, $E(2a, 2b) = \lceil \frac{a}{b} \rceil = 3$ and $I(2a, 2b) = 3 - \frac{a}{b} = q$.

Next, all rational numbers q with $1 < q < 2$ are the inefficiency of some ordered pair (n, k). Again, let $3 - q = \frac{a}{b}$ in lowest terms. The fact that $1 < q < 2$ implies that $1 < \frac{a}{b} < 2$. So by Problem 4a, $E(2a, 2b) = 3$ and $I(2a, 2b) = 3 - \frac{a}{b} = q$.

If $\frac{n}{k} = 2$, then $I(n, k) = 0$ as noted above, while if $\frac{n}{k} > 2$, Problem 6b concludes that $E(n, k) = \lceil \frac{n}{k} \rceil$ or $E(n, k) = \lceil \frac{n}{k} \rceil + 1$. In either case, $I(n, k) \leq \lceil \frac{n}{k} \rceil + 1 - \frac{n}{k} < 2$. As all nonnegative rational numbers less than 2 (except the excluded integer 1) have already been shown to be inefficiencies of ordered pairs, the cases $\frac{n}{k} \geq 2$ can now be excluded from consideration.

Consider ordered pairs (n, k) with $1 < \frac{n}{k} < 2$. If $n + k$ is even, Problem 4a implies that $E(n, k) = 3$ so $I(n, k) = 3 - \frac{n}{k} < 2$ and once again, all such numbers are already known to be inefficiencies. Of course if n is odd and k is even, then $I(n, k)$ is undefined.

Combining these results shows that a rational number will not be the inefficiency of some ordered pair simply whenever it is greater than 2 and avoids being the inefficiency of any ordered pair (n, k) where $1 < \frac{n}{k} < 2$, n is even, and k is odd.

Let $\frac{a}{b} > 2$ be a non-integral rational number in lowest terms. If $\frac{a}{b}$ is the inefficiency of some ordered pair (n, k), then $\frac{a}{b} = E(n, k) - \frac{n}{k}$. Because $E(n, k)$ is an integer, it follows that $\frac{a}{b} + \frac{n}{k}$ is also an integer. Because $1 < \frac{n}{k} < 2$, there is exactly one integer that it can be.

To find non-integral rational numbers that are not the inefficiency of any ordered pair (n, k), choose a rational number $\frac{a}{b}$ in such a way that the only possible choice for $\frac{n}{k}$, in lowest terms, leads to $I(n, k) < \frac{a}{b}$. Then because $I(rn, rk) \leq I(n, k)$, no choice of n and k can possibly lead to $I(n, k) = \frac{a}{b}$.

One such choice is $\frac{a}{b} = \frac{11}{3}$. The only integer between $\frac{11}{3} + 1$ and $\frac{11}{3} + 2$ is $\frac{11}{3} + \frac{4}{3} = 5$. Choosing $n = 4$ and $k = 3$ satisfies the conditions that n is even, k is odd, and $1 < \frac{n}{k} < 2$, and also leads to $I(4, 3) = E(4, 3) - \frac{4}{3} = E(4, 1) - \frac{4}{3} = 4 - \frac{4}{3} = \frac{8}{3} < \frac{11}{3}$. Thus no ordered pair (n, k) in the ratio $\frac{n}{k} = \frac{4}{3}$ can yield an inefficiency of $\frac{11}{3}$.

Another choice that will work is $\frac{a}{b} = \frac{17}{5}$. The only $\frac{n}{k}$ that could yield such an inefficiency would be $\frac{8}{5}$ (again, note that n is even, k is odd, and $1 < \frac{n}{k} < 2$). Then $I(8,5) = E(8,5) - \frac{8}{5} = 4 - \frac{8}{5} = \frac{12}{5} < \frac{17}{5}$, and this is a similar contradiction.

Note that there are many other non-integral rational numbers that are not the inefficiency of any ordered pair. They just aren't covered by the method outlined above and require different techniques. For instance, $\frac{5}{2}$ is such a number. For such an ordered pair (n,k) would require $\frac{n}{k}$ to be an odd multiple of $\frac{1}{2}$. Now n cannot be odd with k even; in that case, $I(n,k)$ is not defined. So both n and k are even. But then previous work shows that $I(n,k) < 2$, so $\frac{5}{2}$ is not an inefficiency of any ordered pair (n,k).

6 Individual Problems

Problem 1. In rectangle $PAUL$, point D is the midpoint of $\overline{UL}$ and points E and F lie on $\overline{PL}$ and $\overline{PA}$, respectively such that $\frac{PE}{EL} = \frac{3}{2}$ and $\frac{PF}{FA} = 2$. Given that $PA = 36$ and $PL = 25$, compute the area of pentagon $AUDEF$.

Problem 2. Rectangle $ARML$ has length 125 and width 8. The rectangle is divided into 1000 squares of area 1 by drawing in gridlines parallel to the sides of $ARML$. Diagonal $\overline{AM}$ passes through the interior of exactly n of the 1000 unit squares. Compute n.

Problem 3. Compute the least integer $n > 1$ such that the product of all positive divisors of n equals n^4.

Problem 4. Each of the six faces of a cube is randomly colored red or blue with equal probability. Compute the probability that no three faces of the same color share a common vertex.

Problem 5. Scalene triangle ABC has perimeter 2019 and integer side lengths. The angle bisector from C meets $\overline{AB}$ at D such that $AD = 229$. Given that AC and AD are relatively prime, compute BC.

Problem 6. Given that a and b are positive and

$$\lfloor 20 - a \rfloor = \lfloor 19 - b \rfloor = \lfloor ab \rfloor,$$

compute the least upper bound of the set of possible values of $a + b$.

Problem 7. Compute the number of five-digit integers $\underline{M}\,\underline{A}\,\underline{R}\,\underline{T}\,\underline{Y}$, with all digits distinct, such that $M > A > R$ and $R < T < Y$.

Problem 8. In parallelogram $ARML$, points P and Q are the midpoints of sides $\overline{RM}$ and $\overline{AL}$, respectively. Point X lies on segment $\overline{PQ}$, and $PX = 3$, $RX = 4$, and $PR = 5$. Point I lies on segment $\overline{RX}$ such that $IA = IL$. Compute the maximum possible value of $\frac{[PQR]}{[LIP]}$.

Problem 9. Given that a, b, c, and d are positive integers such that

$$a! \cdot b! \cdot c! = d! \quad \text{and} \quad a + b + c + d = 37,$$

compute the product $abcd$.

Problem 10. Compute the value of

$$\sin(6^\circ) \cdot \sin(12^\circ) \cdot \sin(24^\circ) \cdot \sin(42^\circ) + \sin(12^\circ) \cdot \sin(24^\circ) \cdot \sin(42^\circ).$$

7 Answers to Individual Problems

Answer 1. 630

Answer 2. 132

Answer 3. 24

Answer 4. $\dfrac{9}{32}$ (or 0.28125)

Answer 5. 888

Answer 6. $\dfrac{41}{5}$ (or $8\frac{1}{5}$ or 8.2)

Answer 7. 1512

Answer 8. $\dfrac{4}{3}$ (or $1\frac{1}{3}$ or $1.\overline{3}$)

Answer 9. 2240

Answer 10. $\dfrac{1}{16}$ (or 0.0625)

8 Solutions to Individual Problems

Problem 1. In rectangle $PAUL$, point D is the midpoint of $\overline{UL}$ and points E and F lie on $\overline{PL}$ and $\overline{PA}$, respectively such that $\dfrac{PE}{EL} = \dfrac{3}{2}$ and $\dfrac{PF}{FA} = 2$. Given that $PA = 36$ and $PL = 25$, compute the area of pentagon $AUDEF$.

Solution 1. For convenience, let $PA = 3x$ and let $PL = 5y$. Then the given equations involving ratios of segment lengths imply that $PE = 3y$, $EL = 2y$, $PF = 2x$, and $FA = x$. Then $[PAUL] = (3x)(5y) = 15xy$ and

$$\begin{aligned}[AUDEF] &= [PAUL] - [PEF] - [ELD]\\ &= 15xy - \frac{1}{2}(3y)(2x) - \frac{1}{2}(2y)\left(\frac{3x}{2}\right)\\ &= 15xy - 3xy - \frac{3xy}{2}\\ &= \frac{21xy}{2}.\end{aligned}$$

Because $15xy = 36 \cdot 25$, it follows that $3xy = 36 \cdot 5 = 180$ and that $\frac{21xy}{2} = \frac{7}{2}(3xy) = \frac{7}{2} \cdot 180 = \mathbf{630}$.

Problem 2. Rectangle $ARML$ has length 125 and width 8. The rectangle is divided into 1000 squares of area 1 by drawing in gridlines parallel to the sides of $ARML$. Diagonal $\overline{AM}$ passes through the interior of exactly n of the 1000 unit squares. Compute n.

Solution 2. Notice that 125 and 8 are relatively prime. Examining rectangles of size $a \times b$ where a and b are small and relatively prime suggests an answer of $a + b - 1$. To see that this is the case, note that other than the endpoints, the diagonal does not pass through any vertex of any unit square. After the first square, it must enter each subsequent square via a vertical or horizontal side. By continuity, the total number of these sides is the sum of the $a - 1$ interior vertical lines and $b - 1$ interior horizontal lines. The diagonal passes through $(a-1)+(b-1) = a+b-2$ additional squares, so the total is $a+b-1$. Because 125 and 8 are relatively prime, it follows that $N = 125 + 8 - 1 = \mathbf{132}$.

Remark: As an exercise, the reader is encouraged to show that the answer for general a and b is $a+b-\gcd(a, b)$.

Problem 3. Compute the least integer $n > 1$ such that the product of all positive divisors of n equals n^4.

Solution 3. Note that every factor pair d and $\frac{n}{d}$ have product n. For the product of all such divisor pairs to equal n^4, there must be exactly 4 divisor pairs, or 8 positive integer divisors. A number has 8 positive integer divisors if it is of the form a^3b^1 or a^7 where a and b are distinct primes. The prime factorization a^3b^1 $(a \neq b)$ provides a set of divisors each of which has 4 options for using a (a^0, a^1, a^2, a^3) and an independent 2 options for using b (b^0, b^1). Using the least values $(a, b) = (2, 3)$, $a^3b^1 = 24$. If instead the prime factorization is a^7 (having divisors $a^0, a^1, a^2, \ldots, a^7$), the least answer would be $2^7 = 128$. Thus the answer is $\mathbf{24}$.

Problem 4. Each of the six faces of a cube is randomly colored red or blue with equal probability. Compute the probability that no three faces of the same color share a common vertex.

Solution 4. There are $2^6 = 64$ colorings of the cube. Let r be the number of faces that are colored red. Define a *monochromatic vertex* to be a vertex of the cube for which the three faces meeting there have the same color. It is clear that a coloring without a monochromatic vertex is only possible in the cases $2 \leq r \leq 4$. If $r = 2$ or $r = 4$, the only colorings that do not have a monochromatic vertex occur when two opposing faces are colored with the minority color (red in the $r = 2$ case, blue in the $r = 4$ case). Because there are 3 pairs of opposite faces of a cube, there are 3 colorings without a monochromatic vertex if $r = 2$ and another 3 such colorings if $r = 4$. For the $r = 3$ colorings, of which there are 20, the only cases in which there *are* monochromatic vertices occur when opposing faces are monochromatic, but in different colors. There are $2^3 = 8$ such colorings, leaving $20 - 8 = 12$ colorings that do not have a monochromatic vertex. Therefore $3 + 3 + 12 = 18$ of the 64 colorings have no monochromatic vertex, and the answer is $\mathbf{\frac{9}{32}}$.

Problem 5. Scalene triangle ABC has perimeter 2019 and integer side lengths. The angle bisector from C meets $\overline{AB}$ at D such that $AD = 229$. Given that AC and AD are relatively prime, compute BC.

Solution 5. Let $BC = a, AC = b, AB = c$. Also, let $AD = e$ and $BD = f$. Then $a + b + e + f = 2019$, the values a, b, and $e + f$ are integers, and by the Angle Bisector Theorem, $\frac{e}{f} = \frac{b}{a}$. So $b = \frac{ae}{f} = \frac{229a}{f}$. Because 229 is prime and $\gcd(b, e) = 1$, conclude that f must be an integer multiple of 229. So let $f = 229x$ for some integer x. Then $a = b \cdot x$ and $a + b + c = 2019$ implies $2019 = bx + b + 229 + 229x = (b + 229)(1 + x)$. Because $2019 = 673 \cdot 3$, it follows that $b = 444$ and $x = 2$, from which $BC = a = \mathbf{888}$.

Problem 6. Given that a and b are positive and

$$\lfloor 20 - a \rfloor = \lfloor 19 - b \rfloor = \lfloor ab \rfloor,$$

compute the least upper bound of the set of possible values of $a + b$.

Solution 6. Let the common value of the three expressions in the given equation be N. Maximizing $a + b$ involves making at least one of a and b somewhat large, which makes the first two expressions for N small. So, to maximize $a + b$, look for the least possible value of N. One can show that $N = 14$ is not possible because that would require $a > 5$ and $b > 4$, which implies $ab > 20$. But $N = 15$ is possible by setting $a = 4 + x, b = 3 + y$, where $0 < x, y \leq 1$. The goal is to find the least upper bound for $x + y$ given $15 \leq (4 + x)(3 + y) < 16 \Rightarrow 3 \leq 3(x + y) + y + xy < 4$. This is equivalent to seeking the maximum value of $x + y$ given $3(x + y) + y + xy \leq 4$. By inspection, if $x = 1$ and $y = \frac{1}{5}$, then $3(x + y) + y + xy = 4 \leq 4$. This is in fact optimal. To see this, consider that because $3x + 3y + y + xy \leq 4$, it follows that $y \leq \frac{4 - 3x}{x + 4}$, and so $x + y \leq x + \frac{4 - 3x}{x + 4} \leq \frac{x^2 + x + 4}{x + 4}$, which is increasing on $0 \leq x \leq 1$. Thus the maximum for $x + y$ is attained when $x = 1$. Hence the least upper bound for $a + b$ is $5 + (3 + \frac{1}{5}) = \mathbf{\frac{41}{5}}$.

Problem 7. Compute the number of five-digit integers $\underline{M}\,\underline{A}\,\underline{R}\,\underline{T}\,\underline{Y}$, with all digits distinct, such that $M > A > R$ and $R < T < Y$.

Solution 7. There are $\binom{10}{5} = 252$ ways to choose the values of the digits M, A, R, T, Y, without restrictions. Because R is fixed as the least of the digits and because $T < Y$, it suffices to find the number of ways to choose M and A. Once M and A are chosen, the other three digits are uniquely determined. There are $\binom{4}{2} = 6$ ways to select M, A. Thus the number of five-digit integers of the type described is $252 \cdot 6 = \mathbf{1512}$.

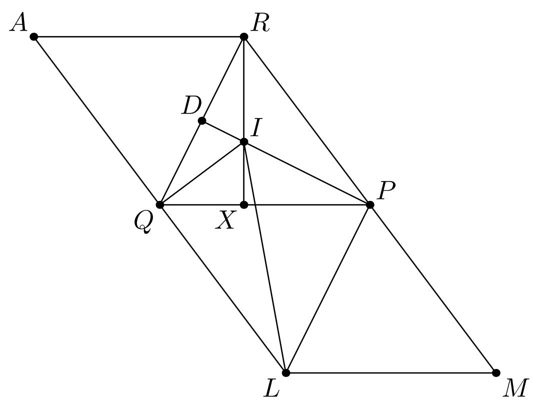
Problem 8. In parallelogram $ARML$, points P and Q are the midpoints of sides $\overline{RM}$ and $\overline{AL}$, respectively. Point X lies on segment $\overline{PQ}$, and $PX = 3$, $RX = 4$, and $PR = 5$. Point I lies on segment $\overline{RX}$ such that $IA = IL$. Compute the maximum possible value of $\frac{[PQR]}{[LIP]}$.

Solution 8. Because $AI = LI$ and $AQ = LQ$, line IQ is the perpendicular bisector of $\overline{AL}$. Because $ARML$ is a parallelogram, $\overline{QI} \perp \overline{RP}$. Note also that $m\angle RXP = 90°$. Thus I is the orthocenter of triangle PQR, from which it follows that $\overleftrightarrow{PI} \perp \overline{RQ}$ and $\overline{PI} \perp \overline{PL}$ (because $PRQL$ is a parallelogram). Extend $\overline{PI}$ through I to meet $\overline{RQ}$ at D. Then $2[PQR] = RQ \cdot PD$ and $2[LIP] = PI \cdot PL = PI \cdot RQ$. Hence the problem is equivalent to determining the maximum value of PD/PI.

Set $\mathrm{m}\angle RPD = \mathrm{m}\angle RPI = \alpha$ and $\mathrm{m}\angle IPX = \beta$, and note that $PD = PR\cos\alpha = 5\cos\alpha$ and $PI = PX/\cos\beta = 3/\cos\beta$. It follows that

$$\frac{PD}{PI} = \frac{5\cos\alpha\cos\beta}{3} = \frac{5\big(\cos(\alpha+\beta)+\cos(\alpha-\beta)\big)}{6} \le \frac{5(3/5+1)}{6} = \frac{\mathbf{4}}{\mathbf{3}},$$

with equality when $\alpha = \beta$.



Problem 9. Given that a, b, c, and d are positive integers such that

$$a! \cdot b! \cdot c! = d! \quad \text{and} \quad a+b+c+d = 37,$$

compute the product $abcd$.

Solution 9. Without loss of generality, assume $a \le b \le c < d$. Note that d cannot be prime, as none of $a!$, $b!$, or $c!$ would have it as a factor. If $d = p+1$ for some prime p, then $c = p$ and $a!b! = p+1$. The least possible values of $a!b!$ are $1, 2, 4, 6, 24, 36, 48, 120, 144, 240$, so the case where $d = p+1$ is impossible. If $d \ge 21$, then $a+b+c \le 16$ and it is impossible to find values of a and b such that $a! \cdot b! = \frac{d!}{c!}$. If $d = 16$, either $a!b! = 16$ or $a!b! = 16 \cdot 15$ or $a!b! = 16 \cdot 15 \cdot 14$. Comparing to the list above, the only possible value $a!b!$ on the list is $16 \cdot 15 = 240$ and so $(a,b,c,d) = (2,5,14,16)$ and $abcd = \mathbf{2240}$.

Problem 10. Compute the value of

$$\sin(6^\circ)\cdot\sin(12^\circ)\cdot\sin(24^\circ)\cdot\sin(42^\circ) + \sin(12^\circ)\cdot\sin(24^\circ)\cdot\sin(42^\circ).$$

Solution 10. Let $S = (1+\sin 6^\circ)(\sin 12^\circ \sin 24^\circ \sin 42^\circ)$. It follows from a sum-to-product identity that $1+\sin 6^\circ = \sin 90^\circ + \sin 6^\circ = 2\sin 48^\circ \cos 42^\circ$. Because the sine of an angle is the cosine of its complement, it follows that

$$S = (2\sin 48^\circ \cos 42^\circ)(\sin 12^\circ \sin 24^\circ \sin 42^\circ) = 2(\sin 48^\circ)^2(\sin 12^\circ \sin 24^\circ \cos 48^\circ).$$

By the double-angle formula, this means $S = \sin 12^\circ \sin 24^\circ \sin 48^\circ \sin 96^\circ$. By a product-to-sum identity,

$$\sin 12^\circ \sin 48^\circ = \frac{\cos 36^\circ - \cos 60^\circ}{2} = \frac{\sqrt{5}-1}{8} \tag{1}$$

and

$$\sin 24^\circ \sin 96^\circ = \frac{\cos 72^\circ - \cos 120^\circ}{2} = \frac{\sqrt{5}+1}{8}. \tag{2}$$

Multiply the expressions on the right-hand sides of (1) and (2) to obtain $\mathbf{\frac{1}{16}}$.

9 Relay Problems

Relay 1-1. Let $a = 19$, $b = 20$, and $c = 21$. Compute

$$\frac{a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}{a + b + c}.$$

Relay 1-2. Let T = TNYWR. Lydia is a professional swimmer and can swim one-fifth of a lap of a pool in an impressive 20.19 seconds, and she swims at a constant rate. Rounded to the nearest integer, compute the number of <u>minutes</u> required for Lydia to swim T laps.

Relay 1-3. Let T = TNYWR. In $\triangle ABC$, $m\angle C = 90^\circ$ and $AC = BC = \sqrt{T-3}$. Circles O and P each have radius r and lie inside $\triangle ABC$. Circle O is tangent to $\overline{AC}$ and $\overline{BC}$. Circle P is externally tangent to circle O and to $\overline{AB}$. Given that points C, O, and P are collinear, compute r.

Relay 2-1. Given that $p = 6.6 \times 10^{-27}$, then $\sqrt{p} = a \times 10^b$, where $1 \leq a < 10$ and b is an integer. Compute $10a + b$ rounded to the nearest integer.

Relay 2-2. Let T = TNYWR. A group of children and adults go to a rodeo. A child's admission ticket costs \$5, and an adult's admission ticket costs more than \$5. The total admission cost for the group is $\$10 \cdot T$. If the number of adults in the group were to increase by 20%, then the total cost would increase by 10%. Compute the number of children in the group.

Relay 2-3. Let T = TNYWR. Rectangles $FAKE$ and $FUNK$ lie in the same plane. Given that $EF = T$, $AF = \frac{4T}{3}$, and $UF = \frac{12}{5}$, compute the area of the intersection of the two rectangles.

10 Relay Answers

Answer 1-1. 60

Answer 1-2. 101

Answer 1-3. $3 - \sqrt{2}$

Answer 2-1. 67

Answer 2-2. 67

Answer 2-3. 262

11 Relay Solutions

Relay 1-1. Let $a = 19$, $b = 20$, and $c = 21$. Compute

$$\frac{a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}{a + b + c}.$$

Solution 1-1. Note that the numerator of the given expression factors as $(a + b + c)^2$, hence the expression to be computed equals $a + b + c = 19 + 20 + 21 = \mathbf{60}$.

Relay 1-2. Let T = TNYWR. Lydia is a professional swimmer and can swim one-fifth of a lap of a pool in an impressive 20.19 seconds, and she swims at a constant rate. Rounded to the nearest integer, compute the number of <u>minutes</u> required for Lydia to swim T laps.

Solution 1-2. Lydia swims a lap in $5 \cdot 20.19 = 100.95$ seconds. The number of minutes required for Lydia to swim T laps is therefore $100.95 \cdot T/60$. With $T = 60$, the desired number of minutes, rounded to the nearest integer, is **101**.

Relay 1-3. Let T = TNYWR. In $\triangle ABC$, $m\angle C = 90^\circ$ and $AC = BC = \sqrt{T-3}$. Circles O and P each have radius r and lie inside $\triangle ABC$. Circle O is tangent to $\overline{AC}$ and $\overline{BC}$. Circle P is externally tangent to circle O and to $\overline{AB}$. Given that points C, O, and P are collinear, compute r.

Solution 1-3. Let A' and B' be the respective feet of the perpendiculars from O to $\overline{AC}$ and $\overline{BC}$. Let H be the foot of the altitude from C to $\overline{AB}$. Because $\triangle ABC$ is isosceles, it follows that $A'OB'C$ is a square, $m\angle B'CO = 45^\circ$, and $m\angle BCH = 45^\circ$. Hence H lies on the same line as C, O, and P. In terms of r, the length CH is $CO + OP + PH = r\sqrt{2} + 2r + r = (3 + \sqrt{2})r$. Because $AC = BC = \sqrt{T-3}$, it follows that $CH = \frac{\sqrt{T-3}}{\sqrt{2}}$. Thus $r = \frac{\sqrt{T-3}}{\sqrt{2}(3+\sqrt{2})} = \frac{(3\sqrt{2}-2)\sqrt{T-3}}{14}$. With $T = 101$, $\sqrt{T-3} = \sqrt{98} = 7\sqrt{2}$, and it follows that $r = \mathbf{3 - \sqrt{2}}$.

Relay 2-1. Given that $p = 6.6 \times 10^{-27}$, then $\sqrt{p} = a \times 10^b$, where $1 \le a < 10$ and b is an integer. Compute $10a + b$ rounded to the nearest integer.

Solution 2-1. Note that $p = 6.6 \times 10^{-27} = 66 \times 10^{-28}$, so $a = \sqrt{66}$ and $b = -14$. Note that $\sqrt{66} > \sqrt{64} = 8$. Because $8.1^2 = 65.61$ and $8.15^2 = 66.4225 > 66$, conclude that $81 < 10\sqrt{66} < 81.5$, hence $10a$ rounded to the nearest integer is 81, and the answer is $81 - 14 = \mathbf{67}$.

Relay 2-2. Let T = TNYWR. A group of children and adults go to a rodeo. A child's admission ticket costs \$5, and an adult's admission ticket costs more than \$5. The total admission cost for the group is $\$10 \cdot T$. If the number of adults in the group were to increase by 20%, then the total cost would increase by 10%. Compute the number of children in the group.

Solution 2-2. Suppose there are x children and y adults in the group and each adult's admission ticket costs $\$a$. The given information implies that $5x + ay = 10T$ and $5x + 1.2ay = 11T$. Subtracting the first equation from the second yields $0.2ay = T \rightarrow ay = 5T$, so from the first equation, $5x = 5T \rightarrow x = T$. With $T = 67$, the answer is **67**.

Relay 2-3. Let T = TNYWR. Rectangles $FAKE$ and $FUNK$ lie in the same plane. Given that $EF = T$, $AF = \frac{4T}{3}$, and $UF = \frac{12}{5}$, compute the area of the intersection of the two rectangles.

Solution 2-3. Without loss of generality, let A, U, and N lie on the same side of $\overline{FK}$. Applying the Pythagorean Theorem to triangle AFK, conclude that $FK = \frac{5T}{3}$. Comparing the altitude to $\overline{FK}$ in triangle AFK to $\overline{UF}$, note that the intersection of the two rectangles will be a triangle with area $\frac{2T^2}{3}$ if $\frac{4T}{5} \leq \frac{12}{5}$, or $T \leq 3$. Otherwise, the intersection will be a trapezoid. In this case, using similarity, the triangular regions of $FUNK$ that lie outside of $FAKE$ each have one leg of length $\frac{12}{5}$ and the others of lengths $\frac{16}{5}$ and $\frac{9}{5}$, respectively. Thus their combined areas $\frac{1}{2} \cdot \frac{12}{5}(\frac{16}{5} + \frac{9}{5}) = 6$, hence the area of the intersection is $\frac{5T}{3} \cdot \frac{12}{5} - 6 = 4T - 6$. With $T = 67$, the answer is therefore **262**.

12 Super Relay

1. Given that a, b, c, and d are integers such that $a + bc = 20$ and $-a + cd = 19$, compute the greatest possible value of c.

2. Let T = TNYWR. Emile randomly chooses a set of T cards from a standard deck of 52 cards. Given that Emile's set contains no clubs, compute the probability that his set contains three aces.

3. Let T = TNYWR. In parallelogram $ABCD$, $\frac{AB}{BC} = T$. Given that M is the midpoint of $\overline{AB}$ and P and Q are the trisection points of $\overline{CD}$, compute $\frac{[ABCD]}{[MPQ]}$.

4. Let T = TNYWR. Compute the value of x such that $\log_T \sqrt{x-7} + \log_{T^2}(x-2) = 1$.

5. Let T = TNYWR. Let p be an odd prime and let x, y, and z be positive integers less than p. When the trinomial $(px + y + z)^{T-1}$ is expanded and simplified, there are N terms, of which M are always multiples of p. Compute M.

6. Let T = TNYWR. Compute the value of K such that 20, $T - 5$, K is an increasing geometric sequence and 19, K, $4T + 11$ is an increasing arithmetic sequence.

7. Let T = TNYWR. Cube $\mathcal{C}_1$ has volume T and sphere $\mathcal{S}_1$ is circumscribed about $\mathcal{C}_1$. For $n \geq 1$, the sphere $\mathcal{S}_n$ is circumscribed about the cube $\mathcal{C}_n$ and is inscribed in the cube $\mathcal{C}_{n+1}$. Let k be the least integer such that the volume of $\mathcal{C}_k$ is at least 2019. Compute the edge length of $\mathcal{C}_k$.

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15. Square $KENT$ has side length 20. Point M lies in the interior of $KENT$ such that $\triangle MEN$ is equilateral. Given that $KM^2 = a - b\sqrt{3}$, where a and b are integers, compute b.

14. Let T = TNYWR. Let a, b, and c be the three solutions of the equation $x^3 - 20x^2 + 19x + T = 0$. Compute $a^2 + b^2 + c^2$.

13. Let T = TNYWR and let $K = \sqrt{T-1}$. Compute $\left|(K-20)(K+1) + 19K - K^2\right|$.

12. Let T = TNYWR. In $\triangle LEO$, $\sin\angle LEO = \frac{1}{T}$. If $LE = \frac{1}{n}$ for some positive real number n, then $EO = n^3 - 4n^2 + 5n$. As n ranges over the positive reals, compute the least possible value of $[LEO]$.

11. Let T = TNYWR. Given that x, y, and z are real numbers such that $x + y = 5$, $x^2 - y^2 = \frac{1}{T}$, and $x - z = -7$, compute $x + z$.

10. Let T = TNYWR. The product of all positive divisors of 2^T can be written in the form 2^K. Compute K.

9. Let T = TNYWR. At the *Westward House of Supper* ("WHS"), a dinner special consists of an appetizer, an entrée, and dessert. There are 7 different appetizers and K different entrées that a guest could order. There are 2 dessert choices, but ordering dessert is optional. Given that there are T possible different orders that could be placed at the WHS, compute K.

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8. Let S be the number you will receive from position 7 and let M be the number you will receive from position 9. Sam and Marty each ride a bicycle at a constant speed. Sam's speed is S km/hr and Marty's speed is M km/hr. Given that Sam and Marty are initially 100 km apart and they begin riding towards one another at the same time, along a straight path, compute the number of kilometers that Sam will have traveled when Sam and Marty meet.

13 Super Relay Answers

1. 39

2. 1

3. 6

4. 11

5. 55

6. 125

7. 15

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15. 400

14. 362

13. 20

12. $\frac{1}{40}$

11. 20

10. 210

9. 10

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8. 60

14 Super Relay Solutions

Problem 1. Given that a, b, c, and d are integers such that $a + bc = 20$ and $-a + cd = 19$, compute the greatest possible value of c.

Solution 1. Adding the two given equations yields $bc + cd = c(b + d) = 39$. The greatest possible value of c therefore occurs when $c = \mathbf{39}$ and $b + d = 1$.

Problem 2. Let $T = \text{TNYWR}$. Emile randomly chooses a set of T cards from a standard deck of 52 cards. Given that Emile's set contains no clubs, compute the probability that his set contains three aces.

Solution 2. Knowing that 13 of the cards are *not* in Emile's set, there are $\binom{39}{T}$ ways for him to have chosen a set of T cards. Given that Emile's set contains no clubs, the suits of the three aces are fixed (i.e., diamonds, hearts, and spades). The number of possible sets of cards in which these three aces appear is therefore $\binom{36}{T-3}$. The desired probability is therefore $\binom{36}{T-3}/\binom{39}{T}$. With $T = 39$, this probability is $1/1 = \mathbf{1}$, which is consistent with the fact that Emile's set contains all cards in the deck that are *not* clubs, hence he is guaranteed to have all three of the remaining aces.

Problem 3. Let $T = \text{TNYWR}$. In parallelogram $ABCD$, $\dfrac{AB}{BC} = T$. Given that M is the midpoint of $\overline{AB}$ and P and Q are the trisection points of $\overline{CD}$, compute $\dfrac{[ABCD]}{[MPQ]}$.

Solution 3. Let $CD = 3x$ and let h be the length of the altitude between bases $\overline{AB}$ and $\overline{CD}$. Then $[ABCD] = 3xh$ and $[MPQ] = \frac{1}{2}xh$. Hence $\dfrac{[ABCD]}{[MPQ]} = \mathbf{6}$. Both the position of M and the ratio $\dfrac{AB}{BC} = T$ are irrelevant.

Problem 4. Let $T = \text{TNYWR}$. Compute the value of x such that $\log_T \sqrt{x-7} + \log_{T^2}(x-2) = 1$.

Solution 4. It can readily be shown that $\log_a b = \log_{a^2} b^2$. Thus it follows that $\log_T \sqrt{x-7} = \log_{T^2}(x-7)$. Hence the left-hand side of the given equation is $\log_{T^2}(x-7)(x-2)$ and the equation is equivalent to $(x-7)(x-2) = T^2$, which is equivalent to $x^2-9x+14-T^2 = 0$. With $T = 6$, this equation is $x^2-9x-22 = 0 \Longrightarrow (x-11)(x+2) = 0$. Plugging $x = -2$ into the given equation leads to the first term of the left-hand side having a negative radicand and the second term having an argument of 0. However, one can easily check that $x = \mathbf{11}$ indeed satisfies the given equation.

Problem 5. Let $T = \text{TNYWR}$. Let p be an odd prime and let x, y, and z be positive integers less than p. When the trinomial $(px + y + z)^{T-1}$ is expanded and simplified, there are N terms, of which M are <u>always</u> multiples of p. Compute M.

Solution 5. A general term in the expansion of $(px + y + z)^{T-1}$ has the form $K(px)^a y^b z^c$, where a, b, and c are nonnegative integers such that $a+b+c = T-1$. Using the "stars and bars" approach, the number of nonnegative integral solutions to $a + b + c = T - 1$ is the number of arrangements of $T - 1$ stars and 2 bars in a row (the bars act has separators and the "2" arises because it is one less than the number of variables in the equation). Thus there are $\binom{T+1}{2}$ solutions. Each term will be a multiple of p <u>unless</u> $a = 0$. In this case, the number of terms that are *not* multiples of p is the number of nonnegative integral solutions to the equation $b + c = T - 1$, which is T (b can range from 0 to $T - 1$ inclusive, and then c is fixed). Hence $M = \binom{T+1}{2} - T = \frac{T^2-T}{2}$. With $T = 11$, the answer is $\mathbf{55}$.

Problem 6. Let $T = \text{TNYWR}$. Compute the value of K such that 20, $T-5$, K is an increasing geometric sequence and 19, K, $4T+11$ is an increasing arithmetic sequence.

Solution 6. The condition that 20, $T-5$, K is an increasing geometric sequence implies that $\frac{T-5}{20} = \frac{K}{T-5}$, hence $K = \frac{(T-5)^2}{20}$. The condition that 19, K, $4T+11$ is an increasing arithmetic sequence implies that $K - 19 = 4T + 11 - K$, hence $K = 2T + 15$. With $T = 55$, each of these equations implies that $K = \mathbf{125}$. Note that the two equations can be combined and solved without being passed a value of T. A quadratic equation results, and its roots are $T = 55$ or $T = -5$. However, with $T = -5$, neither of the given sequences is increasing.

Problem 7. Let $T = \text{TNYWR}$. Cube $\mathcal{C}_1$ has volume T and sphere $\mathcal{S}_1$ is circumscribed about $\mathcal{C}_1$. For $n \geq 1$, the sphere $\mathcal{S}_n$ is circumscribed about the cube $\mathcal{C}_n$ and is inscribed in the cube $\mathcal{C}_{n+1}$. Let k be the least integer such that the volume of $\mathcal{C}_k$ is at least 2019. Compute the edge length of $\mathcal{C}_k$.

Solution 7. In general, let cube $\mathcal{C}_n$ have edge length x. Then the diameter of sphere $\mathcal{S}_n$ is the space diagonal of $\mathcal{C}_n$, which has length $x\sqrt{3}$. This in turn is the edge length of cube $\mathcal{C}_{n+1}$. Hence the edge lengths of $\mathcal{C}_1, \mathcal{C}_2, \ldots$ form an increasing geometric sequence with common ratio $\sqrt{3}$ and volumes of $\mathcal{C}_1, \mathcal{C}_2, \ldots$ form an increasing geometric sequence with common ratio $3\sqrt{3}$. With $T = 125$, the edge length of $\mathcal{C}_1$ is 5, so the sequence of edge lengths of the cubes is $5, 5\sqrt{3}, 15, \ldots$, and the respective sequence of the volumes of the cubes is $125, 375\sqrt{3}, 3375, \ldots$. Hence $k = 3$, and the edge length of $\mathcal{C}_3$ is $\mathbf{15}$.

Problem 15. Square $KENT$ has side length 20. Point M lies in the interior of $KENT$ such that $\triangle MEN$ is equilateral. Given that $KM^2 = a - b\sqrt{3}$, where a and b are integers, compute b.

Solution 15. Let s be the side length of square $KENT$; then $ME = s$. Let J be the foot of the altitude from M to $\overline{KE}$. Then $m\angle JEM = 30^\circ$ and $m\angle EMJ = 60^\circ$. Hence $MJ = \frac{s}{2}$, $JE = \frac{s\sqrt{3}}{2}$, and $KJ = KE - JE = s - \frac{s\sqrt{3}}{2}$. Applying the Pythagorean Theorem to $\triangle KJM$ implies that $KM^2 = \left(s - \frac{s\sqrt{3}}{2}\right)^2 + \left(\frac{s}{2}\right)^2 = 2s^2 - s^2\sqrt{3}$. With $s = 20$, the value of b is therefore $s^2 = \mathbf{400}$.

Problem 14. Let $T = \text{TNYWR}$. Let a, b, and c be the three solutions of the equation $x^3 - 20x^2 + 19x + T = 0$. Compute $a^2 + b^2 + c^2$.

Solution 14. According to Vieta's formulas, $a + b + c = -(-20) = 20$ and $ab + bc + ca = 19$. Noting that $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$, it follows that $a^2 + b^2 + c^2 = 20^2 - 2 \cdot 19 = \mathbf{362}$. The value of T is irrelevant.

Problem 13. Let $T = \text{TNYWR}$ and let $K = \sqrt{T-1}$. Compute $\left|(K-20)(K+1) + 19K - K^2\right|$.

Solution 13. The expression inside the absolute value bars simplifies to $K^2 - 19K - 20 + 19K - K^2 = -20$. Hence the answer is $\mathbf{20}$ and the value of K $(= \sqrt{361} = 19)$ is not needed.

Problem 12. Let $T = \text{TNYWR}$. In $\triangle LEO$, $\sin\angle LEO = \dfrac{1}{T}$. If $LE = \dfrac{1}{n}$ for some positive real number n, then $EO = n^3 - 4n^2 + 5n$. As n ranges over the positive reals, compute the least possible value of $[LEO]$.

Solution 12. Note that $[LEO] = \frac{1}{2}(\sin\angle LEO) \cdot LE \cdot EO = \frac{1}{2} \cdot \frac{1}{T} \cdot \frac{1}{n} \cdot (n^3 - 4n^2 + 5n) = \frac{n^2 - 4n + 5}{2T}$. Because T is a constant, the least possible value of $[LEO]$ is achieved when the function $f(n) = n^2 - 4n + 5$ is minimized.

This occurs when $n = -(-4)/(2 \cdot 1) = 2$, and the minimum value is $f(2) = 1$. Hence the desired least possible value of $[LEO]$ is $\frac{1}{2T}$, and with $T = 20$, this is $\mathbf{\frac{1}{40}}$.

Problem 11. Let $T = $ TNYWR. Given that x, y, and z are real numbers such that $x + y = 5$, $x^2 - y^2 = \dfrac{1}{T}$, and $x - z = -7$, compute $x + z$.

Solution 11. Note that $x^2-y^2 = (x+y)(x-y) = 5(x-y)$, hence $x-y = \frac{1}{5T}$. Then $x+z = (x+y)+(x-y)+(z-x) = 5 + \frac{1}{5T} + 7 = 12 + \frac{1}{5T}$. With $T = \frac{1}{40}$, the answer is thus $12 + 8 = \mathbf{20}$.

Problem 10. Let $T = $ TNYWR. The product of all positive divisors of 2^T can be written in the form 2^K. Compute K.

Solution 10. When n is a nonnegative integer, the product of the positive divisors of 2^n is $2^0 \cdot 2^1 \cdot \ldots \cdot 2^{n-1} \cdot 2^n = 2^{0+1+\cdots+(n-1)+n} = 2^{n(n+1)/2}$. Because $T = 20$ is an integer, it follows that $K = \frac{T(T+1)}{2} = \mathbf{210}$.

Problem 9. Let $T = $ TNYWR. At the *Westward House of Supper* ("WHS"), a dinner special consists of an appetizer, an entrée, and dessert. There are 7 different appetizers and K different entrées that a guest could order. There are 2 dessert choices, but ordering dessert is optional. Given that there are T possible different orders that could be placed at the WHS, compute K.

Solution 9. Because dessert is optional, there are effectively $2+1 = 3$ dessert choices. Hence, by the Multiplication Principle, it follows that $T = 7 \cdot K \cdot 3$, thus $K = \frac{T}{21}$. With $T = 210$, the answer is **10**.

Problem 8. Let S be the number you will receive from position 7 and let M be the number you will receive from position 9. Sam and Marty each ride a bicycle at a constant speed. Sam's speed is S km/hr and Marty's speed is M km/hr. Given that Sam and Marty are initially 100 km apart and they begin riding towards one another at the same time, along a straight path, compute the number of kilometers that Sam will have traveled when Sam and Marty meet.

Solution 8. In km/hr, the combined speed of Sam and Marty is $S + M$. Thus one can determine the total time they traveled and use this to determine the number of kilometers that Sam traveled. However, this is not needed, and there is a simpler approach. Suppose that Marty traveled a distance of d. Then because Sam's speed is $\frac{S}{M}$ of Marty's speed, Sam will have traveled a distance of $\frac{S}{M} \cdot d$. Thus, together, they traveled $d + \frac{S}{M} \cdot d$. Setting this equal to 100 and solving yields $d = \frac{100M}{M+S}$. Thus Sam traveled $\frac{S}{M} \cdot d = \frac{100S}{M+S}$. With $S = 15$ and $M = 10$, this is equal to **60** km.

15 Tiebreaker Problems

Problem 1. Regular tetrahedra $JANE$, $JOHN$, and $JOAN$ have non-overlapping interiors. Compute $\tan \angle HAE$.

Problem 2. Each positive integer less than or equal to 2019 is written on a blank sheet of paper, and each of the digits 0 and 5 is erased. Compute the remainder when the product of the remaining digits on the sheet of paper is divided by 1000.

Problem 3. Compute the third least positive integer n such that each of n, $n+1$, and $n+2$ is a product of exactly two (not necessarily distinct) primes.

16 Tiebreaker Answers

Answer 1. $\dfrac{5\sqrt{2}}{2}$

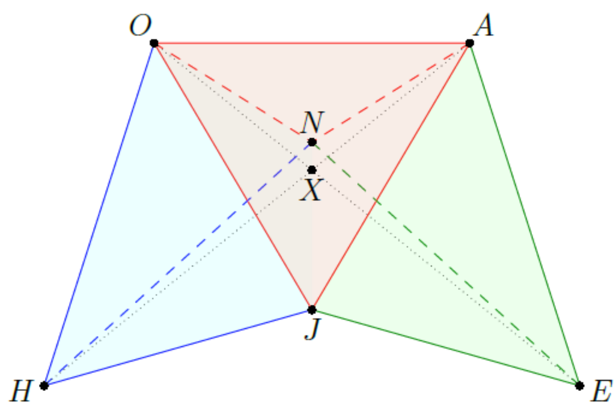
Answer 2. 976

Answer 3. 93

17 Tiebreaker Solutions

Problem 1. Regular tetrahedra $JANE$, $JOHN$, and $JOAN$ have non-overlapping interiors. Compute $\tan \angle HAE$.

Solution 1. First note that $\overline{JN}$ is a shared edge of all three pyramids, and that the viewpoint for the figure below is from along the line that is the extension of edge $\overline{JN}$.



Let h denote the height of each pyramid. Let X be the center of pyramid $JOAN$, and consider the plane passing through H, A, and E. By symmetry, the altitude in pyramid $JOHN$ through H and the altitude in pyramid $JANE$ through E pass through X. Thus points H, X, and A are collinear, as are points E, X, and O. Hence $AH = OE = 2h$. Using the result that the four medians in a tetrahedron are concurrent and divide each other in a $3:1$ ratio, it follows that $AX = OX = \frac{3h}{4}$ and $XE = OE - OX = \frac{5h}{4}$. Applying the Law of Cosines to triangle AXE yields $\cos\angle XAE = \cos\angle HAE = \frac{2-2h^2}{3h}$. Suppose, without loss of generality, that the common side length of the pyramids is 1. Then $h = \sqrt{\frac{2}{3}}$ and $\cos\angle HAE = \frac{\sqrt{6}}{9}$. Hence $\sin\angle HAE = \frac{\sqrt{75}}{9}$ and therefore $\tan\angle HAE = \boldsymbol{\frac{5\sqrt{2}}{2}}$.

Problem 2. Each positive integer less than or equal to 2019 is written on a blank sheet of paper, and each of the digits 0 and 5 is erased. Compute the remainder when the product of the remaining digits on the sheet of paper is divided by 1000.

Solution 2. Count the digits separately by position, noting that 1 is irrelevant to the product. There are a total of 20 instances of the digit 2 in the thousands place. The digit 0 only occurs in the hundreds place if the thousands digit is 2, so look at the numbers 1 through 1999. Each non-zero digit contributes an equal number of times, so there are 200 each of 1, 2, 3, 4, 6, 7, 8, 9. The same applies to the tens digit, except there can be the stray digit of 1 among the numbers 2010 through 2019, but again, these do not affect the product. In the units place, there are 202 of each of the digits. Altogether, there are 602 each of 2, 3, 4, 6, 7, 8, 9, along with 20 extra instances of the digit 2. Note that $9\cdot8\cdot7\cdot6\cdot4\cdot3\cdot2 = 3024\cdot24 = 72{,}576$ leaves a remainder of 576 when divided by 1000. Also $2^{20} = 1024^2 \equiv 24^2 \pmod{1000}$, so 2^{20} contributes another factor of 576. The answer is therefore the remainder when 576^{603} is divided by 1000. This computation can be simplified by using the Chinese Remainder Theorem with moduli 8 and 125, whose product is 1000. Note $576^{603} \equiv 0 \pmod 8$ because 576 is divisible by 8. Also $576 \equiv 76 \pmod{125}$. By Euler's totient theorem, $576^{100} \equiv 1 \pmod{125}$, so $576^{603} \equiv 76^3 \pmod{125}$. This can quickly be computed by noting that $76^3 = (75+1)^3 = 75^3 + 3\cdot75^2 + 3\cdot75 + 1 \equiv 3\cdot75+1 \equiv -24 \pmod{125}$. Observing that $-24 \equiv 0 \pmod 8$, it follows that $576^{603} \equiv -24 \pmod{1000}$, hence the desired remainder is **976**.

Problem 3. Compute the third least positive integer n such that each of n, $n+1$, and $n+2$ is a product of exactly two (not necessarily distinct) primes.

Solution 3. Define a positive integer n to be a *semiprime* if it is a product of exactly two (not necessarily distinct) primes. Define a *lucky trio* to be a sequence of three consecutive integers, $n, n+1, n+2$, each of which is a semiprime. Note that a lucky trio must contain exactly one multiple of 3. Also note that the middle number in a lucky trio must be even. To see this, note that if the first and last numbers in a lucky trio were both even, then exactly one of these numbers would be a multiple of 4. But neither $2, 3, 4$ nor $4, 5, 6$ is a lucky trio, and if a list of three consecutive integers contains a multiple of 4 that is greater than 4, this number cannot be a semiprime. Using this conclusion and because $3, 4, 5$ is not a lucky trio, it follows that the middle number of a lucky trio cannot be a multiple of 4. Hence it is necessary that a lucky trio has the form $4k+1, 4k+2, 4k+3$, for some positive integer k, with $2k+1$ being a prime. Note that $k \not\equiv 1 \pmod 3$ because when $k = 1$, the sequence $5, 6, 7$ is not a lucky trio, and when $k > 1$, $4k+2$ would be a multiple of 6 greater than 6, hence it cannot be a semiprime. Trying $k = 2, 3, 5, 6, 8, 9, \ldots$ allows one to eliminate sequences of three consecutive integers that are not lucky trios, and if lucky trios are ordered by their least elements, one finds that the first three lucky trios are $33, 34, 35$; $85, 86, 87$; and $93, 94, 95$. Hence the answer is **93**.