ARML Competition 2018

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1 Team Problems

Problem 1. Compute the number of non-empty subsets S of $\{1, 2, 3, \ldots, 20\}$ for which $|S| \cdot \max\{S\} = 18$. (Note: $|S|$ is the number of elements of the set S.)

Problem 2. A class of 218 students takes a test. Each student's score is an integer from 0 to 100, inclusive. Compute the greatest possible difference between the mean and the median scores.

Problem 3. Regular hexagon $RANGES$ has side length 6. Pentagon $RANGE$ is revolved 360° about the line containing $\overline{RE}$ to obtain a solid. The volume of the solid is $k \cdot \pi$. Compute k.

Problem 4. A fair 12-sided die has faces numbered 1 through 12. The die is rolled twice, and the results of the two rolls are x and y, respectively. Given that $\tan(2\theta) = \frac{x}{y}$ for some θ with $0 < \theta < \frac{\pi}{2}$, compute the probability that $\tan\theta$ is rational.

Problem 5. The absolute values of the five complex roots of $z^5 - 5z^2 + 53 = 0$ all lie between the positive integers a and b, where $a < b$ and $b - a$ is minimal. Compute the ordered pair (a, b).

Problem 6. Let S be the set of points (x, y) whose coordinates satisfy the system of equations:

$$\lfloor x \rfloor \cdot \lceil y \rceil = 20$$
$$\lceil x \rceil \cdot \lfloor y \rfloor = 18.$$

Compute the least upper bound of the set of distances between points in S.

Problem 7. Compute the least integer $d > 0$ for which there exist distinct lattice points A, B, and C in the coordinate plane, each exactly $\sqrt{d}$ units from the origin, satisfying $\csc(\angle ABC) > 2018$.

Problem 8. Compute the number of unordered collections of three integer-area rectangles such that the three rectangles can be assembled without overlap to form one 3×5 rectangle. (For example, one such collection contains one 3×3 and two 1×3 rectangles, and another such collection contains one 3×3 and two 2×1.5 rectangles. The latter collection is equivalent to the collection of two 1.5×2 rectangles and one 3×3 rectangle.)

Problem 9. Let Γ be a circle with diameter $\overline{XY}$ and center O, and let γ be a circle with diameter $\overline{OY}$. Circle ω_1 passes through Y and intersects Γ and γ again at A and B, respectively. Circle ω_2 also passes through Y and intersects Γ and γ again at D and C, respectively. Given that $AB = 1$, $BC = 4$, $CD = 2$, and $AD = 7$, compute the sum of the areas of ω_1 and ω_2.

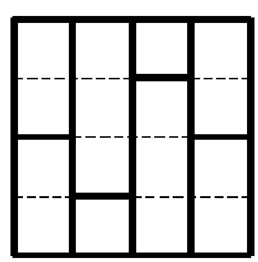
Problem 10. In the number puzzle below, clues are given for the four rows, each of which contains a four-digit number. Cells inside a region bounded by bold lines must all contain the same digit, and each of the eight regions contains a different digit. The variables in the clues are all positive integers. Complete the number puzzle.

1: $4^a + 13^b + 14^c$

2: $5^p + 13^q + 17^r$

3: $4^x + 5^y + 31^z$

4: the average of the other 3 rows



2 Answers to Team Problems

Answer 1. 19

Answer 2. $\dfrac{5400}{109}$ $\left(\text{or } 49\dfrac{59}{109}\right)$

Answer 3. $342\sqrt{3}$

Answer 4. $\dfrac{1}{18}$ (or $0.0\bar{5}$)

Answer 5. $(2, 3)$

Answer 6. $2\sqrt{73}$

Answer 7. 2,038,181

Answer 8. 132

Answer 9. 20π

Answer 10.

5	1	9	7
5	1	0	7
4	1	0	2
4	8	0	2

3 Solutions to Team Problems

Problem 1. Compute the number of non-empty subsets S of $\{1, 2, 3, \ldots, 20\}$ for which $|S| \cdot \max\{S\} = 18$. (Note: $|S|$ is the number of elements of the set S.)

Solution 1. Both $|S|$ and $\max\{S\}$ are integers between 1 and 20, so factor 18 and consider the possible ways to have $|S|$ and $\max\{S\}$ equal each factor. Then count the number of possible sets S that satisfy that condition.

$\lvert S\rvert$	$\max\{S\}$	**Number of possible sets S**
1	18	1: this can only be the set $S = \{18\}$.
2	9	8: this can be any set $S = \{a, 9\}$, where $1 \le a \le 8$.
3	6	$\binom{5}{2} = 10$: the greatest element must be 6, and there are $\binom{5}{2}$ ways of selecting the two other elements of S.
6	3	0: this is impossible because S cannot have 6 elements, the greatest of which is 3.
9	2	0: this is impossible because S cannot have 9 elements, the greatest of which is 2.
18	1	0: this is impossible because S cannot have 18 elements, the greatest of which is 1.

Thus the answer is $1 + 8 + 10 = \mathbf{19}$.

Problem 2. A class of 218 students takes a test. Each student's score is an integer from 0 to 100, inclusive. Compute the greatest possible difference between the mean and the median scores.

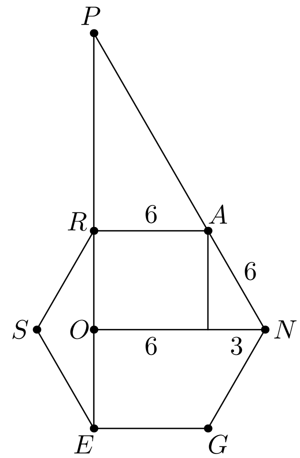
Solution 2. Intuitively, the maximum difference can be attained by forcing the median to be 0 by selecting just enough 0 scores, and then maximizing the mean by maximizing the remaining scores. Such a distribution would have 110 scores of 0 and 108 scores of 100, giving a median score of 0 and a mean score of $\frac{108 \cdot 100}{218} = \frac{5400}{109}$. This gives a difference of $\frac{5400}{109}$.

To prove that this is maximal, assume that the median score is m. Then, by definition, at least 109 scores must be at most m. Further, the remaining 109 scores are at most 100, so the mean μ satisfies $\mu \le \frac{1}{218}(109m + 10900) = \frac{m+100}{2}$. Therefore $\mu - m \le \frac{100-m}{2}$.

If $m = 0$, then the maximum mean is found as above. If $m \ge 1$, then $\mu - m \le \frac{100-1}{2} = 49.5 < \frac{5400}{109}$, and the difference is greater than it is in the above case with $m = 0$. There only remains the possibility that $m = \frac{1}{2}$. For this to be the case, 109 scores must be 0 and one score must be 1. To maximize the mean, the remaining 108 scores must be 100. However, compared to the case of having 110 scores of 0 and 108 of 100, this increases the median by $\frac{1}{2}$ but increases the mean by only 1218, so the difference will certainly be smaller. Thus the answer is $\mathbf{\frac{5400}{109}}$.

Problem 3. Regular hexagon $RANGES$ has side length 6. Pentagon $RANGE$ is revolved 360° about the line containing $\overline{RE}$ to obtain a solid. The volume of the solid is $k \cdot \pi$. Compute k.

Solution 3. First note that because $\overline{RA}$ and $\overline{EG}$ are parallel, the solid obtained will consist of two conic frustums, each with a base along the plane through $\overline{NS}$ and perpendicular to $\overline{ER}$. It is easier to compute the volume of one of these conic frustums at a time. Let O be the midpoint of $\overline{ER}$, and let $\overrightarrow{NA}$ and $\overrightarrow{ER}$ intersect at P. Then the conic frustum is a cone with radius ON and height OP, cut off by the plane containing $\overline{AR}$ and perpendicular to $\overline{ER}$.



Note that $ON = NS - OS = 12 - 3 = 9$, and $OP = OR + RP = 3\sqrt{3} + 6\sqrt{3} = 9\sqrt{3}$. Then the volume of one of the conic frustums is

$$\frac{1}{3}\pi\left(9^2\right)\left(9\sqrt{3}\right) - \frac{1}{3}\pi\left(6^2\right)\left(6\sqrt{3}\right) = 171\sqrt{3}\,\pi.$$

The volume of the entire solid is therefore $342\sqrt{3}\cdot\pi$, so $k = \mathbf{342\sqrt{3}}$.

Problem 4. A fair 12-sided die has faces numbered 1 through 12. The die is rolled twice, and the results of the two rolls are x and y, respectively. Given that $\tan(2\theta) = \frac{x}{y}$ for some θ with $0 < \theta < \frac{\pi}{2}$, compute the probability that $\tan\theta$ is rational.

Solution 4. Suppose $\tan 2\theta = \frac{x}{y}$. Then the identity for the tangent of a sum gives $\frac{2\tan\theta}{1-\tan^2\theta} = \frac{x}{y}$, which is equivalent to $x - x\tan^2\theta = 2y\tan\theta$. Rearranging gives a quadratic equation in $\tan\theta$: $x\tan^2\theta + 2y\tan\theta - x = 0$. Thus $\tan\theta$ is rational if and only if the discriminant is a perfect square. The discriminant is $(2y)^2 - 4\cdot x\cdot(-x) = 4(x^2+y^2)$, which is a perfect square if and only if $x^2 + y^2$ is a perfect square. The only Pythagorean triples (a, b, c) with $a, b \le 12$ are $(3, 4, 5)$, $(6, 8, 10)$, $(9, 12, 15)$, and $(5, 12, 13)$. Therefore the only (x, y) that give a rational value for $\tan\theta$ are $(3, 4), (4, 3), (6, 8), (8, 6), (9, 12), (12, 9), (5, 12), (12, 5)$. The desired probability is therefore $\frac{8}{12\cdot 12} = \mathbf{\frac{1}{18}}$.

Problem 5. The absolute values of the five complex roots of $z^5 - 5z^2 + 53 = 0$ all lie between the positive integers a and b, where $a < b$ and $b - a$ is minimal. Compute the ordered pair (a, b).

Solution 5. First note that the question asks for the tightest possible integer bounds on the magnitude of a zero of the complex polynomial $z^5 - 5z^2 + 53$, essentially describing the smallest complex annulus with integer radii that contains the roots.

Now recall two important properties of the absolute value function: $|ab| = |a||b|$ and $|a + b| \le |a| + |b|$ for all complex numbers a and b. The latter also implies that $|a| - |b| \le |a - b|$. In order to have $z^5 - 5z^2 + 53 = 0$, it must be the case that $|z^5 - 5z^2| = 53$. This is enough to restrict the possible magnitude of z.

Suppose that $|z| > 3$. Then $|z^5 - 5z^2| = |z|^2|z^3 - 5| > 9|z^3 - 5| \ge 9(|z|^3 - 5) > 9(27 - 5) = 198 > 53$. So $|z| \le 3$.

Suppose that $|z| < 2$. Then $|z^5 - 5z^2| = |z|^2|z^3 - 5| < 4|z^3 - 5| \le 4(|z|^3 + |5|) < 4(8 + 5) = 51 < 53$. So $|z| \ge 2$.

Therefore $2 \le |z| \le 3$ for all roots z of $z^5 - 5z^2 + 53$. Because $a < b$, and a and b are integers, $b - a$ is at least 1, so this is the best possible bound. The answer is $\mathbf{(2, 3)}$.

Problem 6. Let S be the set of points (x, y) whose coordinates satisfy the system of equations:

$$\lfloor x \rfloor \cdot \lceil y \rceil = 20$$
$$\lceil x \rceil \cdot \lfloor y \rfloor = 18.$$

Compute the least upper bound of the set of distances between points in S.

Solution 6. First note that x and y cannot both be integers because otherwise, the given system would be inconsistent. Now suppose that exactly one of x or y is an integer. If x is an integer and $\lfloor y \rfloor = b$, then $\lceil y \rceil = b + 1$, and multiplying the two given equations and simplifying results in the equation $x^2 \cdot b(b+1) = 360$ (†). Because $1 \le x^2 < 360$, the only possible values of x^2 are the perfect square factors of 360, namely 1, 4, 9, and 36. Substituting in $x^2 = 1, 9,$ and 36 into (†) results in non-integral solutions for b. But substituting $x^2 = 4$ into (†) results in $b^2 + b - 90 = (b + 10)(b - 9) = 0$, and $b = -10$ or $b = 9$. If $x = 2$, then $\lfloor y \rfloor = 9$ and $\lceil y \rceil = 10$, and this yields the solutions $(2, y)$, where $9 < y < 10$. On the other hand, if $x = -2$, then $\lfloor y \rfloor = -9$ and $\lceil y \rceil = -10$, which is impossible. By a similar argument, if x is not an integer and y is an integer, this results in the solutions $(x, -2)$, where $-10 < x < -9$. Thus if $\mathcal{I}_1$ is the open interval $(9, 10)$ and $\mathcal{I}_2$ is the open interval $(-10, -9)$, then the solutions (x, y) to the given system where exactly one of x and y is an integer are the ordered pairs belonging to the union of the two sets $\{2\} \times \mathcal{I}_1$* and $\mathcal{I}_2 \times \{-2\}$. Graphically, these represent segments of unit length (one vertical, one horizontal) that do not include the endpoints. For later reference, let $\mathcal{I}_x = \mathcal{I}_2 \times \{-2\}$ and let $\mathcal{I}_y = \{2\} \times \mathcal{I}_1$.

Now suppose that neither x nor y is an integer and that $\lfloor x \rfloor = a$ and $\lfloor y \rfloor = b$. Then $\lceil x \rceil = a + 1, \lceil y \rceil = b + 1$, and multiplying the two given equations and simplifying results in the equation $a(a+1)b(b+1) = 360$ (‡). By noting the factorization $360 = 3 \cdot 4 \cdot 5 \cdot 6$, the following ordered pairs (a, b) satisfy (‡):

$$(3, 5); \quad (-4, 5); \quad (-4, -6); \quad (3, -6); \quad (5, 3); \quad (5, -4); \quad (-6, -4); \quad (-6, 3).$$

However, only $(a, b) = (5, 3)$ and $(-4, -6)$ satisfy the given system:

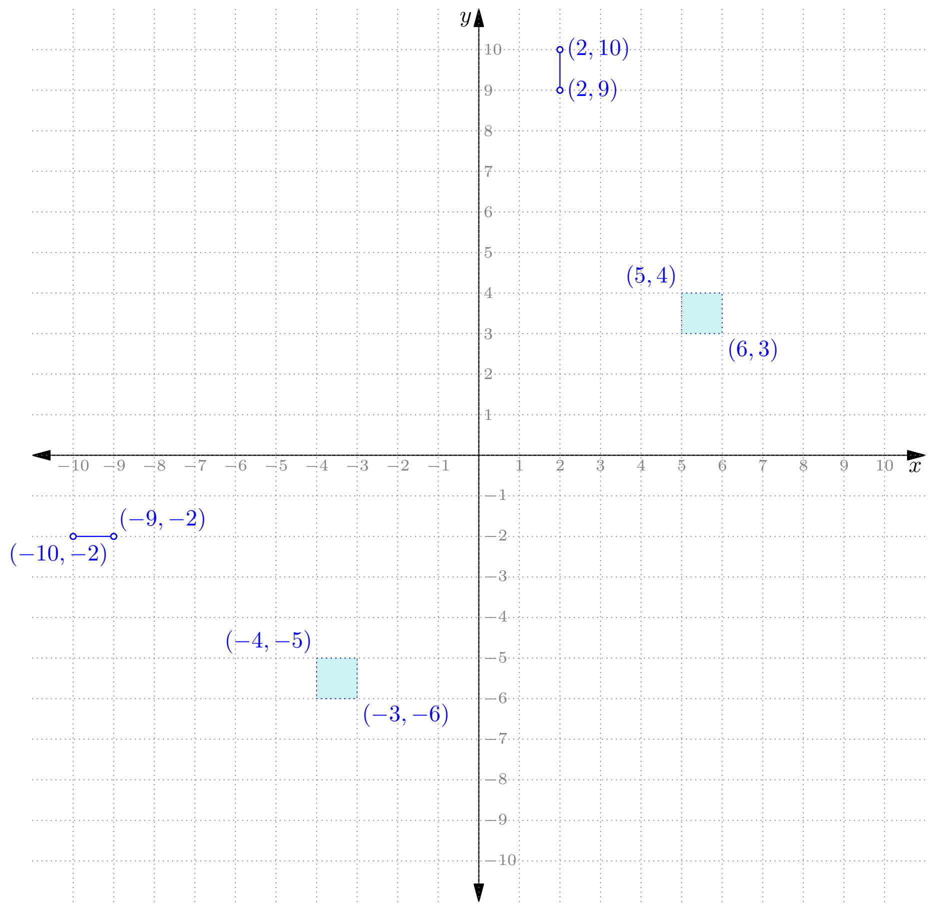
$$a(b + 1) = 20$$
$$(a + 1)b = 18.$$

Let $\mathcal{I}_3$ and $\mathcal{I}_4$ denote the open intervals $(5, 6)$ and $(3, 4)$, respectively and let $\mathcal{I}_5$ and $\mathcal{I}_6$ denote the open intervals $(-4, -3)$ and $(-6, -5)$, respectively. Then the solutions (x, y) to the given system in which neither x nor y is an integer are given by the union of the two regions $\mathcal{R}_1$ and $\mathcal{R}_2$, defined by

$$\mathcal{R}_1 = \mathcal{I}_3 \times \mathcal{I}_4 \quad \text{and} \quad \mathcal{R}_2 = \mathcal{I}_5 \times \mathcal{I}_6.$$

Each of $\mathcal{R}_1$ and $\mathcal{R}_2$ is the interior of a unit square ($\mathcal{R}_1$ lies in the first quadrant and $\mathcal{R}_2$ lies in the third quadrant). Also note that the solutions are symmetric about the line $y = -x$. A plot of the points of S is shown.

*Here, $\times$ denotes the Cartesian product. I.e., if P and Q are two sets, then $P \times Q = \{(p, q) \mid p \in P \text{ and } q \in Q\}$.



The answer to the problem is $\max\{\ell_1, \ell_2, \ell_3\}$, where:

- ℓ_1 is the least upper bound of the distances between a point in $\mathcal{I}_x$ and a point in $\mathcal{I}_y$;
- ℓ_2 is the least upper bound of the distances between a point in $\mathcal{R}_1$ and a point in $\mathcal{R}_2$;
- ℓ_3 is the least upper bound of the distances between a point in $\mathcal{I}_x \cup \mathcal{I}_y$ and a point in $\mathcal{R}_1 \cup \mathcal{R}_2$.

Compute ℓ_1 by taking the distance between the extremal boundary points of $\mathcal{I}_x$ and $\mathcal{I}_y$, namely $(-10, -2)$ and $(2, 10)$, respectively. This distance is $\sqrt{2 \cdot 12^2} = \sqrt{288}$.

Compute ℓ_2 by taking the distance between the extremal boundary points of $\mathcal{R}_1$ and $\mathcal{R}_2$, namely $(6, 4)$ and $(-4, -6)$, respectively. This distance is $\sqrt{2 \cdot 10^2} = \sqrt{200}$.

Compute ℓ_3 by taking the distance between the extremal boundary points of $\mathcal{I}_x$ and $\mathcal{R}_1$ (or of $\mathcal{I}_y$ and $\mathcal{R}_2$, owing to the symmetry), namely $(-10, -2)$ and $(6, 4)$, respectively. This distance is $\sqrt{16^2 + 6^2} = \sqrt{292}$.

Thus the answer is $\max\{\sqrt{288}, \sqrt{200}, \sqrt{292}\} = \sqrt{292} = \mathbf{2\sqrt{73}}$.

Problem 7. Compute the least integer $d > 0$ for which there exist distinct lattice points A, B, and C in the coordinate plane, each exactly $\sqrt{d}$ units from the origin, satisfying $\csc(\angle ABC) > 2018$.

Solution 7. The points A, B, and C lie on a circle with diameter $2\sqrt{d}$. Consequently, by the Extended Law of Sines,

$$2\sqrt{d} = \frac{AC}{\sin(\angle ABC)} \rightarrow \csc(\angle ABC) = \frac{2\sqrt{d}}{AC}.$$

The solution now proceeds in three cases.

- It is impossible that $AC = 1$. Indeed, if $A = (x, y)$, then without loss of generality, assume $C = (x+1, y)$. Then $d = x^2 + y^2 = (x+1)^2 + y^2$, which implies $2x + 1 = 0$, contradiction.
- Suppose $AC = \sqrt{2}$. If $A = (x, y)$, then without loss of generality, assume $C = (x+1, y+1)$, by reflecting or rotating as necessary. Then $d = x^2 + y^2 = (x+1)^2 + (y+1)^2$, hence $2x + 1 + 2y + 1 = 0$. It follows that $y = -(x+1)$ and d must be of the form $d = x^2 + (x+1)^2$. In that case,

$$\csc(\angle ABC) = \frac{2\sqrt{d}}{\sqrt{2}} = \sqrt{2(x^2 + (x+1)^2)}$$

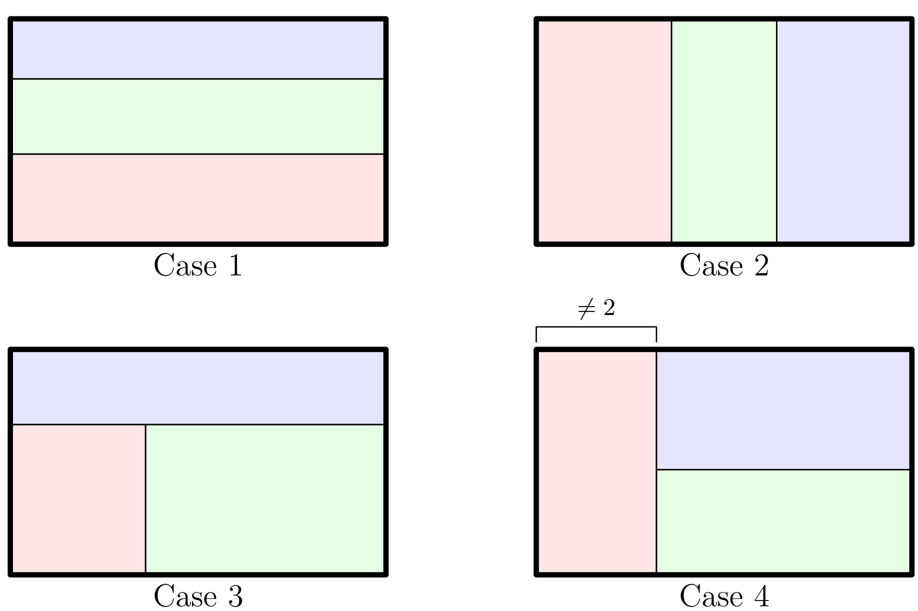
  and the least x for which this exceeds 2018 is $x = 1009$, meaning $d = 1009^2 + 1010^2$.
- Finally assume $AC \geq \sqrt{3}$. Then $2018 < \frac{2\sqrt{d}}{AC} \leq \sqrt{\frac{4}{3}d}$, which would mean $d > 2018^2 \cdot \frac{3}{4} > 1009^2 + 1010^2$. Thus all values of d achieved in this case are greater than the value of d in the second case.

In conclusion, the least possible value of d is $1009^2 + 1010^2 = \mathbf{2038181}$. This value of d is indeed possible, as the points $A = (1009, 1010)$, $B = (-1009, -1010)$, and $C = (1010, 1009)$ satisfy the conditions of the problem.

Problem 8. Compute the number of unordered collections of three integer-area rectangles such that the three rectangles can be assembled without overlap to form one 3×5 rectangle. (For example, one such collection contains one 3×3 and two 1×3 rectangles, and another such collection contains one 3×3 and two 2×1.5 rectangles. The latter collection is equivalent to the collection of two 1.5×2 rectangles and one 3×3 rectangle.)

Solution 8. In order to enumerate the possible rectangles it is necessary to consider several cases. One possible approach is presented below.

Consider the possible dissections of the given 3×5 rectangle. There are four shapes which can be achieved, as shown below.



Begin by counting the number of unordered collections that arise from each case.

- In Case 1, a collection corresponds to a multiset $\{a, b, c\}$ satisfying $a + b + c = 15$ (as the three rectangles can be reordered freely). Assume, without loss of generality, that $a \leq b \leq c$. To eliminate the inequality constraints, let $x = a - 1$, $y = b - a$, $z = c - b$ denote nonnegative integers. Accordingly, $a = x + 1$, $b = x + y + 1$, $z = x + y + z + 1$, and the given equation now reads $x + 2y + 3z = 12$ or, equivalently, $2y + 3z \leq 12$. The number of solutions in nonnegative integers is then

$$\sum_{z=0}^{4}\left(1 + \left\lfloor \frac{12 - 3z}{2} \right\rfloor\right) = 7 + 5 + 4 + 2 + 1 = 19.$$

- Case 2 has 19 solutions by repeating the argument from Case 1 verbatim.
- In Case 3, denote by $15 - n$ the area of the top rectangle. The number of possible dissections of the lower rectangle is then exactly $\lfloor n/2 \rfloor$. Moreover, the top rectangle (because it has a side length of 5) is never congruent to either of the lower rectangles. Thus the number of solutions in this case is

$$\sum_{n=1}^{14}\left\lfloor \frac{n}{2} \right\rfloor = 49.$$

- In Case 4, assume the leftmost region does not have width 2. This ensures that Case 2 and Case 4 will be disjoint from each other, because the two rectangles on the right will not form a 3×3 square that could be rotated to obtain a dissection already counted in Case 2.

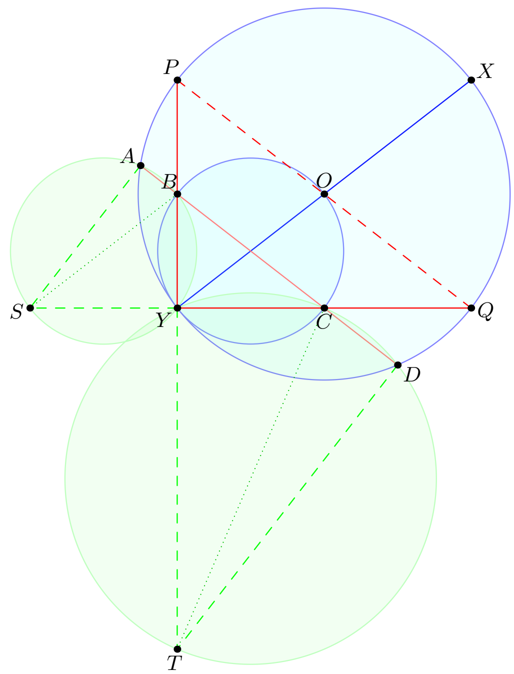
  In order to count the number of such configurations, denote by $15 - n$ the area of the left rectangle, where $15 - n \neq 3 \cdot 2$ or, equivalently, $n \neq 9$. Then repeating the logic of Case 3 gives a count of

$$\sum_{\substack{n=1 \\ n\neq 9}}^{14}\left\lfloor \frac{n}{2} \right\rfloor = 49 - \left\lfloor \frac{9}{2} \right\rfloor = 45.$$

It has already been checked that Cases 3 and 4 are disjoint, and the other pairs of cases are seen to be mutually exclusive by comparing the number of sides of length 5 in the dissection. Consequently, the final answer is $19 + 19 + 49 + 45 = \mathbf{132}$.

Problem 9. Let Γ be a circle with diameter $\overline{XY}$ and center O, and let γ be a circle with diameter $\overline{OY}$. Circle ω_1 passes through Y and intersects Γ and γ again at A and B, respectively. Circle ω_2 also passes through Y and intersects Γ and γ again at D and C, respectively. Given that $AB = 1$, $BC = 4$, $CD = 2$, and $AD = 7$, compute the sum of the areas of ω_1 and ω_2.

Solution 9. Note that A, B, C, and D are collinear, in that order, because $AD = AB + BC + CD$. Extend $\overline{YB}$ and $\overline{YC}$ to meet Γ again at P and Q, respectively. Then $\overline{BC}$ is the midline of $\triangle YPQ$, by homothety. By power of a point, $AB \cdot BD = 1 \cdot (4+2) = PB \cdot BY = BY^2$, so $BY = \sqrt{6}$. Similarly, $CY = \sqrt{2 \cdot (4+1)} = \sqrt{10}$. In particular, $\triangle BYC$ is right.



Now let S and T be the antipodes of B and C on ω_1 and ω_2, respectively. It remains to evaluate BS and CT. First note that $m\angle BYS = 90°$ and $m\angle CAS = m\angle BAS = 90°$. Because $m\angle BYC = 90°$, it follows that points C, Y, and S are collinear and that $\triangle CYB \sim \triangle CAS$. Therefore

$$AS = \frac{YB}{YC} \cdot AC = \sqrt{\frac{6}{10}} \cdot 5 = \sqrt{15},$$

hence $BS = \sqrt{AS^2 + AB^2} = 4$. In the same fashion, with $\triangle BYC \sim \triangle BDT$,

$$DT = \frac{YC}{BY} \cdot BD = \sqrt{\frac{10}{6}} \cdot 6 = \sqrt{60},$$

hence $CT = \sqrt{DT^2 + CD^2} = 8$. So the sum of the areas of ω_1 and ω_2 is $\pi(2^2 + 4^2) = \boldsymbol{20\pi}$.

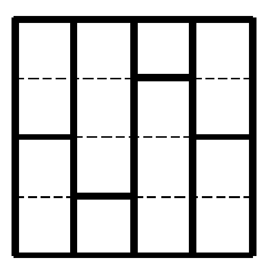
Problem 10. In the number puzzle below, clues are given for the four rows, each of which contains a four-digit number. Cells inside a region bounded by bold lines must all contain the same digit, and each of the eight regions contains a different digit. The variables in the clues are all positive integers. Complete the number puzzle.

1: $4^a + 13^b + 14^c$

2: $5^p + 13^q + 17^r$

3: $4^x + 5^y + 31^z$

4: the average of the other 3 rows



Solution 10. The following solution only relies on the clues for the third and fourth rows. Label the digits in the array as follows.

A	B	C	D
A	B	Y	D
W	B	Y	Z
W	X	Y	Z

The solution proceeds in three steps.

Step 1. The final row is the average of the first three rows, which implies

$$3(1000W + 100X + 10Y + Z) = 1000(2A + W) + 100(3B) + 10(C + 2Y) + (2D + Z).$$

This rearranges to

$$0 = 2000(A - W) + 300(B - X) + 10(C - Y) + 2(D - Z).$$

By comparing the magnitudes of $|2000(A-W)|$ to $|300(B-X)+10(C-Y)+2(D-Z)|$, it is impossible to have $|A - W| \geq 2$, hence $A - W = \pm 1$. A similar argument gives $B - X = \mp 7$, and then $C - Y = \pm 9$, $D - Z = \pm 5$, where the signs correspond. In particular, $\{C, Y\} = \{0, 9\}$.

Step 2. Next, the number in the third row satisfies

$$4^x + 5^y + 31^z \equiv 4^x + 6 \pmod{10}$$

and consequently either $Z = 0$ or $Z = 2$. However, one of C and Y must be 0, hence $\boxed{Z = 2}$. This can only occur if $\boxed{D = 7}$, which determines all the signs in the previous step: it follows that $A - W = -1$, $B - X = -7$, $C - Y = 9$, $D - Z = 5$. In particular, $\boxed{C = 9}$ and $\boxed{Y = 0}$.

Step 3. Because $B - X = -7$ and the digits 0 and 2 have already been used, it follows that $\boxed{B = 1}$ and $\boxed{X = 8}$. The third clue now reads

$$4^x + 5^y + 31^z = \underline{W}\ \underline{1}\ \underline{0}\ \underline{2}.$$

The mod 10 calculation in the previous step implies that x is even and in particular, $x > 1$. Taking the previous equation modulo 8 gives $5^y + 31^z \equiv 102 \equiv 6 \pmod{8}$, which can only occur if y is odd and z is even. This implies $z = 2$ (as $z \leq 2$). Therefore $4^x + 5^y \equiv 141 \pmod{1000}$ and because $x \leq 6$ is even, $y \leq 5$, and this forces $x = 2$ and $y \in \{3, 5\}$. The value $y = 3$ would give $W = 1$, which is not permitted because $B = 1$, so $y = 5$ and $\boxed{W = 4}$. Finally, $\boxed{A = 5}$.

The completed number puzzle reads as follows.

5	1	9	7
5	1	0	7
4	1	0	2
4	8	0	2

4 Power Question 2018: Partitions

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

Binary Partitions

A *binary partition* of a positive integer n is an ordered n-tuple of non-increasing integers, each of which is either 0 or a power of 2, whose sum is n. Each of the integers in the n-tuple is called a *part* of the partition. Each binary partition of n has n parts. Let $p_2(n)$ denote the number of binary partitions of n. For example, $p_2(3) = 2$ because of the two ordered triples $(2, 1, 0)$ and $(1, 1, 1)$.

1. Compute $p_2(n)$ for $n = 4$, 5, 6, and 7. [4 pts]

2. **a.** Show that $p_2(n) \leq p_2(n + 1)$ for all positive integers n. [3 pts]

b. Is the inequality strict for sufficiently large n? Justify your answer. [3 pts]

3. **a.** Prove that if n is even and $n \geq 4$, then $p_2(n) = p_2\left(\frac{n}{2}\right) + p_2(n - 2)$. [2 pts]

b. Find the least $n > 1$ such that $p_2(n)$ is odd, or prove that no such n exists. [2 pts]

Partial Orderings

A *partial ordering* on a set S is a relation, usually denoted $\preceq$, such that all of the following conditions are true:

- $a \preceq a$ for all $a \in S$ **(reflexivity property)**,
- $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ for all $a, b, c \in S$ **(transitivity property)**, and
- $a \preceq b$ and $b \preceq a$ implies $a = b$ for all $a, b \in S$ **(antisymmetry property)**.

The word "partial" refers to the possibility that some two elements, a and b, may be *incomparable*, i.e., neither $a \preceq b$ nor $b \preceq a$ (these negative relations on $\preceq$ are sometimes written as $a \npreceq b$ and $b \npreceq a$, respectively). This Power Question largely defines and explores a partial ordering on the set of binary partitions of n.

The notation $a \prec b$ means that $a \preceq b$ and $a \neq b$. The symbols $\succ$ and $\succeq$ may also be used, and they are defined in the following way: $a \succ b$ means $b \prec a$, and $a \succeq b$ means $b \preceq a$.

If a and b are elements of S with $a \prec b$, and if there is no element $c \in S$ for which $a \prec c \prec b$, then it is said that b *covers* a.

Let a denote the binary partition $(a_1, a_2, \ldots, a_n)$. Similarly, let b denote the binary partition $(b_1, b_2, \ldots, b_n)$. In this Power Question, define $a \prec b$ if it is possible to obtain a from b by a sequence of replacing one 2^k by two 2^{k-1}s (and deleting a 0). For example, $(4, 1, 1, 1, 1, 0, 0, 0) \prec (4, 4, 0, 0, 0, 0, 0, 0)$ because $(4, 1, 1, 1, 1, 0, 0, 0) \prec (4, 2, 1, 1, 0, 0, 0, 0) \prec (4, 2, 2, 0, 0, 0, 0, 0) \prec (4, 4, 0, 0, 0, 0, 0, 0)$. This Power Question will use this partial ordering on binary partitions.

4. **a.** Show that the binary partitions of 5 are totally ordered; i.e., if p and p' are two different binary partitions of 5, then either $p \prec p'$ or $p' \prec p$. [2 pts]

b. Show that the binary partitions of 8 are not totally ordered, i.e., find two binary partitions of 8 – call them q and q' – such that $q \not\prec q'$ and $q' \not\prec q$. [3 pts]

5. **a.** Find the smallest binary partition of n, using this partial ordering. That is, find the binary partition p such that for all other binary partitions p', $p \prec p'$. [2 pts]

b. Find the largest binary partition of n, using this partial ordering. That is, find the binary partition P such that for all other binary partitions P', $P \succ P'$. [3 pts]

Hasse Diagrams

Suppose that a set S has a partial ordering $\preceq$. Then a *Hasse diagram* can be used to display the covering relation in a graphical way. The Hasse diagram is a graph whose vertices are the elements of S and where edges are drawn between two elements x and y if $x \prec y$ or $y \prec x$ and if there is no element z for which $x \prec z$ and $z \prec y$ or for which $y \prec z$ and $z \prec x$. Also, y appears "above" x if $x \prec y$. Note that it is possible for more than one element to appear on the same level of a Hasse diagram. For example, the partially ordered set of divisors of 12, ordered by divisibility, is shown in Figure 1. In this diagram, the number 1 is said to be at Level 0 because it is the least divisor in the partial ordering shown. The numbers 2 and 3 are said to be on Level 1 because $2 \succ 1$ and $3 \succ 1$ and there is no number n such that $2 \succ n$ and $n \succ 1$ or $3 \succ n$ and $n \succ 1$. Other Levels are similarly defined. The number 12 is on Level 3.

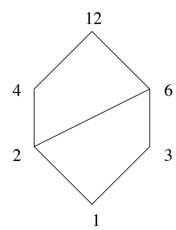


Figure 1

Hasse diagrams can be drawn to show the ordering of partitions such as the ones from Problems 4 and 5. For convenience, rather than labeling the vertices in the Hasse diagram with the partition itself, like $(8, 0, 0, 0, 0, 0, 0, 0)$, it is common to label the vertex with its nonzero parts only, using exponents to indicate parts within the partition with multiplicity greater than 1. For example, the partition $(8, 0, 0, 0, 0, 0, 0, 0)$ would be labeled 8, the partition $(4, 2, 2, 0, 0, 0, 0, 0)$ would be labeled 42^2, and the partition $(2, 2, 1, 1, 1, 1, 0, 0)$ would be labeled $2^2 1^4$.

6. **a.** Draw the Hasse diagram for the binary partitions of 8. Label each vertex. [1 pt]

b. List a path through the Hasse diagram for the binary partitions of 8 that begins at the bottom vertex (that is, the vertex at Level 0) and, traveling only along edges, passes through every other vertex exactly once. Such a path is called a *Hamiltonian path*. [1 pt]

c. Let n be an even integer with $n > 4$. Let S be the set of binary partitions of n. Let S_1 be the set of binary partitions of $\frac{n}{2}$. Let S_2 be the set of binary partitions of $n - 2$. Prove that there is a bijection B (i.e., a one-to-one correspondence) from the set S to the set $S_1 \cup S_2$. Prove that this bijection B preserves order; that is, given that $p \prec p'$ for binary partitions $p, p' \in S$, then either $B(p) \prec B(p')$ or $B(p)$ and $B(p')$ are incomparable. [3 pts]

d. Prove that for each positive integer n, the Hasse diagram of the binary partitions of n has a Hamiltonian path that begins with the vertex at Level 0. [5 pts]

Let $f_L(n)$ represent the number of elements at level L in the Hasse diagram of the binary partitions of n.

7. Prove that the value of $f_L(n)$ is the number of binary partitions of n that have $n - L$ nonzero parts. [3 pts]

8. Not all partitions are binary partitions. Some partitions have parts of the form $2^j - 1$ where j is a nonnegative integer. Such partitions will be called *s-partitions*, and their parts are written in nonincreasing order. Two s-partitions of 5 are $(3, 1, 1, 0, 0)$ and $(1, 1, 1, 1, 1)$.

a. List two partitions of 7, one that is an s-partition and one that is neither an s-partition nor a binary partition. Make sure to identify which is which. [2 pts]

b. Prove that if $n \geq 2L$, the value of $f_L(n)$ is equal to the number of s-partitions of L. [3 pts]

Trinary Partitions

A *trinary partition* of a positive integer n is an ordered n-tuple of non-increasing integers, each of which is either 0 or a power of 3, whose sum is n. Let $p_3(n)$ denote the number of trinary partitions of n. For example, $p_3(4) = 2$ because of the two ordered quadruples $(3, 1, 0, 0)$ and $(1, 1, 1, 1)$.

As with binary partitions, one can define partial orderings for trinary partitions. Let c denote the trinary partition $(c_1, c_2, \ldots, c_n)$. Similarly, let d denote the trinary partition $(d_1, d_2, \ldots, d_n)$. Define $c \prec d$ if it is possible to obtain c from d by a sequence of replacing one 3^k by three 3^{k-1}s (and deleting two 0s).

9. Draw the Hasse diagram for the trinary partitions of 12. Label each vertex. [2 pts]

10. State a value of n less than 23 for which the Hasse diagram of the trinary partitions of n does **not** contain a Hamiltonian path. Prove your claim. (Recall that a Hamiltonian path is defined in Problem 6b.) [6 pts]

5 Solutions to Power Question

1. The values are as follows.
The value of $p_2(4)$ is **4** from $(4,0,0,0)$, $(2,2,0,0)$, $(2,1,1,0)$, and $(1,1,1,1)$.
The value of $p_2(5)$ is **4** from $(4,1,0,0,0)$, $(2,2,1,0,0)$, $(2,1,1,1,0)$, and $(1,1,1,1,1)$.
The value of $p_2(6)$ is **6** from $(4,2,0,0,0,0)$, $(4,1,1,0,0,0)$, $(2,2,2,0,0,0)$, $(2,2,1,1,0,0)$, $(2,1,1,1,1,0)$, and $(1,1,1,1,1,1)$.
The value of $p_2(7)$ is **6** from $(4,2,1,0,0,0,0)$, $(4,1,1,1,0,0,0)$, $(2,2,2,1,0,0,0)$, $(2,2,1,1,1,0,0)$, $(2,1,1,1,1,1,0)$, and $(1,1,1,1,1,1,1)$.

2. a. A binary partition for n can be changed into a binary partition for $n+1$ by inserting a 1 immediately following the rightmost nonzero entry. For example, $(2,2,1,1,0,0)$ is a binary partition of 6, and this can be changed into $(2,2,1,1,1,0,0)$, which is a binary partition of 7. Similarly, $(1,1,1,1)$ becomes $(1,1,1,1,1)$, and these are binary partitions of 4 and 5, respectively. Different binary partitions of n become different binary partitions of $n+1$ in this way. Because every binary partition of n maps to a binary partition of $n+1$, there are at least as many binary partitions of $n+1$ as of n.

b. The answer to the question is **no**. If n is even, then $n+1$ is odd. Because a binary partition of an odd number must contain at least one 1, the correspondence described in the solution to Problem 2a is one-to-one. To establish this, given a binary partition of $n+1$, delete one 1 to obtain a binary partition of n. Thus $p_2(n+1) = p_2(n)$ for any even n. However, note that if n is odd, then $p_2(n+1)$ is strictly greater than $p_2(n)$ because in addition to the binary partitions of $n+1$ that can be obtained by inserting a 1 into a binary partition of n, there are binary partitions of $n+1$ consisting of all even numbers.

3. a. Let n be an even integer. Given a binary partition of n, either all of its parts are even or there is at least one 1. If all parts are even, then all of the positive parts must occur in the first $\frac{n}{2}$ elements of the n-tuple (otherwise, their sum would exceed n). Dividing all the parts by 2 yields an n-tuple of powers of 2 and 0s that sum to $\frac{n}{2}$. Note that the last $\frac{n}{2}$ elements of this n-tuple must all be 0s and truncating them results in an $\left(\frac{n}{2}\right)$-tuple which is a binary partition of $\frac{n}{2}$. Conversely, for any binary partition of $\frac{n}{2}$, by doubling each element and appending $\frac{n}{2}$ zeros at the end results in an n-tuple which is a binary partition of n. Hence $p_2\left(\frac{n}{2}\right)$ counts the number of binary partitions of n in which all the parts are even. If there is at least one 1, then the assumption that n is even means that there are at least two 1s, and deleting them produces a binary partition of $n-2$. Conversely, for each binary partition of $n-2$, appending two 1s (right before the 0s, if there are any, otherwise, append them to the end of the $(n-2)$-tuple) produces an n-tuple that is a binary partition of n. Hence $p_2(n-2)$ counts the number of binary partitions of n with at least one 1, and thus the desired recursion follows.

b. **There is no such *n*.** Proceed by cases. If n is odd, then $p_2(n) = p_2(n-1)$ because all binary partitions of an odd number include at least one 1, and removing that 1 gives a bijection to partitions of $n-1$. If n is even, then $p_2(n) = p_2\left(\frac{n}{2}\right) + p_2(n-2)$ by Problem 3a. Because $p_2(2) = 2$ and $p_2(4) = 4$, it follows inductively that $p_2(n)$ is even for all $n > 1$.

4. a. The four binary partitions of 5 are ordered in the following way: $(1,1,1,1,1) \prec (2,1,1,1,0) \prec (2,2,1,0,0) \prec (4,1,0,0,0)$. This is a total ordering.

b. Consider these two binary partitions of 8: $(4,1,1,1,1,0,0,0)$ and $(2,2,2,2,0,0,0,0)$. The 4 can break down into two 2s, but that is the only way to generate 2s, and that does not generate four 2s, so it is not the case that $(2,2,2,2,0,0,0,0) \prec (4,1,1,1,1,0,0,0)$. Likewise, it is not possible for a partition with no 4s to be related by $\prec$ to a partition with 4s, so it is not the case that $(4,1,1,1,1,0,0,0) \prec (2,2,2,2,0,0,0,0)$.

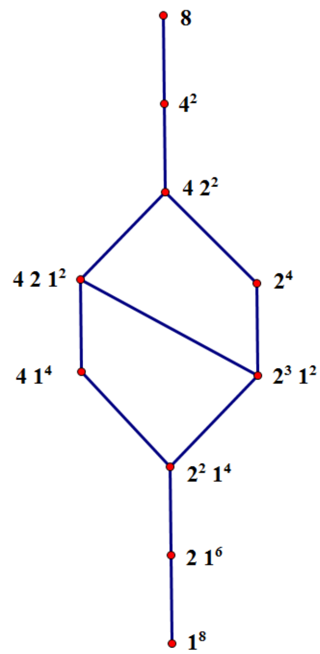
Similarly, the following two pairs of partitions are not comparable:

$$(4,1,1,1,1,0,0,0) \text{ and } (2,2,2,1,1,0,0,0), \qquad (4,2,1,1,0,0,0,0) \text{ and } (2,2,2,2,0,0,0,0).$$

5. a. The smallest binary partition of n is the partition with n 1s. That is, it is the partition $(1,1,\ldots,1)$ with n copies of 1 in the partition. If a partition has any part that is not equal to 0 or 1, then that part is a positive power of 2 that may be replaced by lesser powers of 2. This process can continue until all of the powers of 2 are 1s, but the process cannot continue beyond that because a partition must consist only of integers. Thus every binary partition is greater than the one consisting of all 1s. Also note that the partition $(1, 1, \ldots, 1)$ is not larger than or incomparable with any other partition. Thus $(1, 1, \ldots, 1)$ is the smallest binary partition.

b. The largest binary partition of n is the one that essentially gives the base-2 equivalent of n. That is, if the base-2 representation of n has 1s in the 2^a, 2^b, 2^c places, and so on, with $a > b > c > \cdots$, then the largest binary partition of n is $(2^a, 2^b, 2^c, \ldots, 0, 0, 0)$ (with enough 0s to finish the partition). If a partition contains two copies of a number other than 0, they may be combined to form a greater power of 2. Thus, starting with any partition, continue combining powers of 2 until the partition contains no repeated numbers. Because no further combinations are allowed at this point, by the definition, there can be no partition larger than a partition with no repeated numbers. *A priori* there could be other partitions that are not comparable to such a partition, however. But because such a partition expresses n as a sum of powers of 2, with no repeated nonzero parts, it is essentially the base-2 representation of n, which is unique. So because any partition may undergo the process of combining like powers of 2 until there are no repeats, every binary partition is less than the unique partition which expresses n in binary.

6. **a.** A Hasse diagram for the binary partitions of 8 is shown below.



b. The following sequence of vertices is a Hamiltonian path: $1^8, 21^6, 2^21^4, 41^4, 421^2, 2^31^2, 2^4, 42^2, 4^2, 8$. This path begins at the vertex at Level 0, then visits each vertex exactly once.

c. The bijection was effectively established in the solution to 3a. Now consider whether $B(p) \prec B(p')$ for p and p' which are different binary partitions of n and for which $p \prec p'$. Suppose first that both p and p' have no 1s at all. Then dividing all parts by 2 preserves the covering relation. This is because p can be produced by swapping out powers of 2 in p' for lesser powers of 2, and dividing all parts by 2 does not change this covering. Now suppose that p and p' both have at least two 1s. Then removing two 1s from each partition preserves the covering relation. This is because removing two 1s from the partition in no way changes the "swap-outs" of powers of 2 for lesser powers of 2. Now suppose that one of p or p' has two 1s and the other doesn't. Then $B(p)$ is incomparable with $B(p')$, because $B(p) \in S_1$ and $B(p') \in S_2$ or vice versa, and these are sets of partitions of different numbers because $n - 2 > \frac{n}{2}$ when $n > 4$. The relation $\prec$ is not defined between partitions of different numbers, so $B(p)$ and $B(p')$ are not comparable.

d. As in the solutions to Problems 2 and 3, it is only necessary to consider the problem for even n. This is because a Hasse diagram for partitions of $2r + 1$ is the same as that for the partitions of $2r$ with an extra 1 in every partition. So let $n = 2r = a \cdot 2^k$, where a is an odd integer and k is positive. The proof proceeds by strong induction to prove that there is a Hamiltonian path through the Hasse diagram for partitions of n starting at 1^n and ending at $(2^k)^a$. To get started, the path $1^2 \to 2$ is clearly a Hamiltonian path through the vertices of the Hasse diagram for partitions of 2, and the path $1^4 \to 21^2 \to 2^2 \to 4$ is a Hamiltonian path through the vertices of the Hasse diagram for partitions of 4. These establish the base case.

The idea of the proof will be to use the bijection from part c. In essence, the partitions of n will be split into those that are matched with partitions of $n-2$ and those that are partitions of $n/2$, and the Hamiltonian paths through each of these sets of smaller partitions will be joined into a single path through the partitions of n.

The path starts at 1^n. Consider this as $1^{n-2}1^2$. That is, group the first $n-2$ 1s together and the last two 1s together. From the induction hypothesis, there is a Hamiltonian path through the partitions of $n-2$ starting with 1^{n-2}. By adjoining two 1s to each of these, there is a Hamiltonian path through the partitions of n that have two or more 1s in them. This path ends at $(2^\ell)^c 1^2$, where $n-2 = 2(r-1) = c \cdot 2^\ell$, c is an odd integer, and ℓ is positive. The manner in which the path will be continued through the partitions of n with no 1s in them is dependent on the parity of r.

First consider the case in which r is even. Then $r-1$ is odd. In this case, $c = r-1$ and $\ell = 1$. Thus the path in the previous paragraph ended at $2^c 1^2$. The next step in the path will be to $2^{c+1} = 2^r$. Now by induction, there is a Hamiltonian path through the partitions of $n/2 = r$ starting at 1^r and ending at $(2^{k-1})^a$. Double every entry of each of these partitions. This gives a Hamiltonian path through the partitions of n that have no 1s in them, starting at 2^r and ending at $(2^k)^a$. Attaching this to the end of the path from 1^n to 2^r already obtained results in the desired Hamiltonian path from 1^n to $(2^k)^a$.

In the case where r is odd, $r-1$ is even. In this case, $a = r$ and $k = 1$, while $n - 2 = 2(r-1)$ must be a multiple of 4. So the path through the partitions of n containing 1s ended at $(2^\ell)^c 1^2$ for some $\ell \geq 2$. The next step will be to the partition $(2^\ell)^c 2$. Now $r = n/2$ is odd, so every binary partition of r must contain at least one 1. Note that these are the binary partitions of $r-1$ with a 1 adjoined, and by induction, there is a Hamiltonian path through them starting at 1^r and ending at $(2^{\ell-1})^c 1$. Double each entry of each of these, and run through this path *backward* to obtain a path starting at $(2^\ell)^c 2$ and ending at 2^r. Concatenating this with the path from 1^n to $(2^\ell)^c 2$ previously obtained will yield the desired path from 1^n to $2^r = (2^k)^a$.

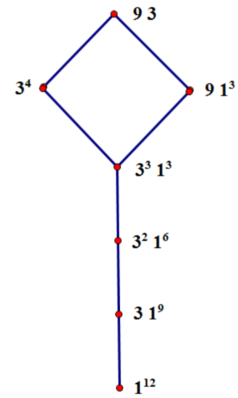
7. Proceed by induction on L. The only element for which $L = 0$ is the smallest binary partition of n, which indeed has n 1s by the solution to Problem 5, and so this partition has $n - L = n - 0$ parts. Moving up via a covering relation, when two 2^ks are replaced by one 2^{k+1}, it is true that one nonzero part is lost. Assuming an element at level L has $n - L$ nonzero parts, an element at level $L+1$ will have $n - L - 1 = n - (L+1)$ parts. This establishes the inductive step.

8. **a.** There are several s-partitions of 7. They include $(7,0,0,0,0,0,0)$, $(3,3,1,0,0,0,0)$, $(3,1,1,1,1,0,0)$, and $(1,1,1,1,1,1,1)$. Other partitions of 7 are not s-partitions; some of those are also not binary. They include $(6,1,0,0,0,0,0)$, $(5,2,0,0,0,0,0)$, $(5,1,1,0,0,0,0)$, $(3,2,2,0,0,0,0)$, and $(3,2,1,1,0,0,0)$.

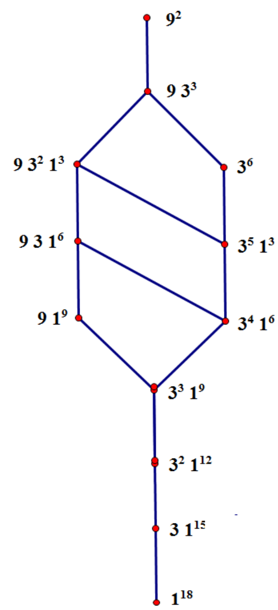
b. Construct a bijection. Given a binary partition of n at level L, subtract 1 from each nonzero part. Hence each part 2^j is replaced by $2^j - 1$, leaving an s-partition of $n - (n - L) = L$.

For the inverse map, consider an s-partition of L, with parts of the form $2^j - 1$ for some positive integer j. This cannot have more than L parts, and that is achievable only with all parts equal to 1. Consider the partition to end in enough 0s to bring the total number of parts up to $n - L$; this is possible because $n - L = n - 2L + L \geq 2L - 2L + L = L$. Now add 1 to each part. The result is a binary partition of $L + n - L = n$ with $n - L$ parts, which puts it at level L by Problem 7.

9. A Hasse diagram for the trinary partitions of 12 is shown below.



10. The three possible answers are **18** or **19** or **20**. A Hasse diagram for the trinary partitions of 18 is shown below. The answer will be established when it is shown that there is no Hamiltonian path through the Hasse diagram. Note that all Hasse diagrams for the trinary partitions of 18 are isomorphic to the one in the figure. Note also that the Hasse diagram for the trinary partitions of 19 is isomorphic to the Hasse diagram for the trinary partitions of 18; a bijection can be established by adding a 1 after the rightmost nonzero part of any trinary partition of 18. Similarly, the Hasse diagram for the trinary partitions of 20 is isomorphic to the Hasse diagram for the trinary partitions of 18.



Suppose there is a Hamiltonian path through the Hasse diagram. Because the Hasse diagram has only one edge coming from 1^{18} and only one edge coming from 9^2, the Hamiltonian path must begin and end at those vertices in some order. Without loss of generality, assume the path begins at 1^{18} and ends at 9^2.

The path must be of the form $1^{18}, 31^{15}, 3^21^{12}, 3^31^9, \ldots, 9^2$. The first choice comes when deciding if the fifth vertex in the path is 91^9 or 3^41^6. If the choice is to make 3^41^6 the fifth vertex in the path, then it is not possible to enter and leave 91^9 because the edge $\{3^31^9, 91^9\}$ cannot be used for entry or for exit; both would require revisiting 3^31^9. So the fifth vertex in the path must be 91^9, and thus the sixth vertex in the path is 931^6.

The next choice comes when considering the seventh vertex in the path. If the seventh vertex is 93^21^3, then it will be impossible to visit 3^41^6 and also 9^2. If the seventh vertex is 3^41^6, then the eighth vertex in the path is 3^51^3, and either choice from that vertex (93^21^3 or 3^6) leads to a contradiction because to visit the other requires a path that prevents visiting 9^2. This completes the proof that the Hasse diagram of the trinary partitions of 18 does not contain a Hamiltonian path.

Checking other possible values of n that are multiples of 3, there is a Hamiltonian path through the Hasse diagrams for each one, as the following argument demonstrates.

- A Hamiltonian path through the Hasse diagram for $n = 3$ is $1^3, 3$.
- A Hamiltonian path through the Hasse diagram for $n = 6$ is $1^6, 31^3, 3^2$.
- A Hamiltonian path through the Hasse diagram for $n = 9$ is $1^9, 31^6, 3^21^3, 3^3, 9$.
- A Hamiltonian path through the Hasse diagram for $n = 12$ is $1^{12}, 31^9, 3^21^6, 3^31^3, 91^3, 93, 3^4$.
- A Hamiltonian path through the Hasse diagram for $n = 15$ is $1^{15}, 31^{12}, 3^21^9, 3^31^6, 91^6, 931^3, 93^2, 3^5, 3^41^3$.
- A Hamiltonian path through the Hasse diagram for $n = 21$ is $1^{21}, 31^{18}, 3^21^{15}, 3^31^{12}, 91^{12}, 931^9, 3^41^9, 3^51^6, 93^21^6, 93^31^3, 93^4, 9^23, 9^21^3$.

Note that there are Hamiltonian paths through the Hasse diagram for both $n = 1$ (the path is just the vertex 1) and $n = 2$ (the path is $1^2, 2$). Note also that if k is a positive integer, the Hasse diagrams for the trinary partitions of $n = 3k$, $n = 3k + 1$, and $n = 3k + 2$ are all isomorphic, in the same way that the Hasse diagrams for the trinary partitions of 18, 19, and 20 are isomorphic.

Thus the only values of n less than 23 for which the Hasse diagram of trinary partitions does not contain a Hamiltonian path are 18 or 19 or 20.

6 Individual Problems

Problem 1. Compute the greatest prime factor of N, where

$$N = 2018 \cdot 517 + 517 \cdot 2812 + 2812 \cdot 666 + 666 \cdot 2018.$$

Problem 2. Compute the number of ordered pairs of positive integers (a, b) that satisfy

$$a^2 b^3 = 20^{18}.$$

Problem 3. Compute the number of positive three-digit multiples of 3 whose digits are distinct and nonzero.

Problem 4. In $\triangle ABC$, $\cos(A) = \dfrac{2}{3}$ and $\cos(B) = \dfrac{2}{7}$. Given that the perimeter of $\triangle ABC$ is 24, compute the area of $\triangle ABC$.

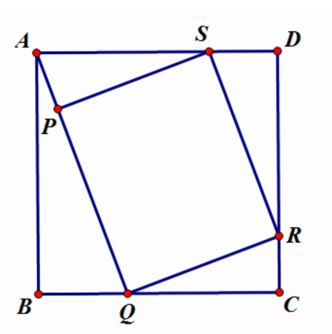
Problem 5. Compute the product of all positive values of x that satisfy

$$\lfloor x+1 \rfloor^{2x} - 19\lfloor x+1 \rfloor^{x} + 48 = 0.$$

Problem 6. Triangle ABC is inscribed in circle ω. The line containing the median from A meets ω again at M, the line containing the angle bisector of $\angle B$ meets ω again at R, and the line containing the altitude from C meets ω again at L. Given that quadrilateral $ARML$ is a rectangle, compute the degree measure of $\angle BAC$.

Problem 7. Let S be the set of points (x, y) whose coordinates satisfy $x^2 + y^2 \leq 36$ and $(\max\{x, y\})^2 \leq 27$. Compute the perimeter of S.

Problem 8. Rectangle $PQRS$ is drawn inside square $ABCD$, as shown. Given that $[APS] = 20$ and $[CQR] = 18$, compute $[ABCD]$.



Problem 9. Compute the least positive four-digit integer N for which N and $N + 2018$ contain a total of 8 distinct digits.

Problem 10. Compute the least positive value of t such that

$$\operatorname{Arcsin}(\sin(\alpha)),\ \operatorname{Arcsin}(\sin(2\alpha)),\ \operatorname{Arcsin}(\sin(7\alpha)),\ \operatorname{Arcsin}(\sin(t\alpha))$$

is a geometric progression for some α with $0 < \alpha < \frac{\pi}{2}$.

7 Answers to Individual Problems

Answer 1. 23

Answer 2. 28

Answer 3. 180

Answer 4. $12\sqrt{5}$

Answer 5. 4

Answer 6. 36 (or 36°)

Answer 7. $8\pi + 12$ (or $12 + 8\pi$)

Answer 8. 243

Answer 9. 1489

Answer 10. $9 - 4\sqrt{5}$ (or $-4\sqrt{5} + 9$)

8 Solutions to Individual Problems

Problem 1. Compute the greatest prime factor of N, where

$$N = 2018 \cdot 517 + 517 \cdot 2812 + 2812 \cdot 666 + 666 \cdot 2018.$$

Solution 1. Factor N as follows:

$$\begin{aligned} N &= 517 \cdot (2018 + 2812) + 666 \cdot (2812 + 2018) \\ &= (517 + 666) \cdot (2018 + 2812) \\ &= 1183 \cdot 4830 \\ &= (7 \cdot 169) \cdot (2 \cdot 5 \cdot 3 \cdot 161) \\ &= (7 \cdot 13^2) \cdot (2 \cdot 3 \cdot 5 \cdot 7 \cdot 23). \end{aligned}$$

Each of the integers 2, 3, 5, 7, 13, and 23 is prime. Thus the largest prime factor of N is **23**.

Problem 2. Compute the number of ordered pairs of positive integers (a, b) that satisfy

$$a^2 b^3 = 20^{18}.$$

Solution 2. The prime factorization of 20^{18} is $2^{36} \cdot 5^{18}$, so a and b may only have prime factors of 2 and 5. Let $a = 2^x \cdot 5^y$. Then $b^3 = \frac{2^{36} \cdot 5^{18}}{a^2} = 2^{36-2x} \cdot 5^{18-2y}$. The value of b will be an integer if and only if both $36 - 2x$ and $18 - 2y$ are nonnegative multiples of 3. Therefore x and y must both be nonnegative multiples of 3. This means that $x \in \{0, 3, 6, 9, 12, 15, 18\}$ and $y \in \{0, 3, 6, 9\}$, so there are a total of $7 \cdot 4 = \mathbf{28}$ solution pairs.

Problem 3. Compute the number of positive three-digit multiples of 3 whose digits are distinct and nonzero.

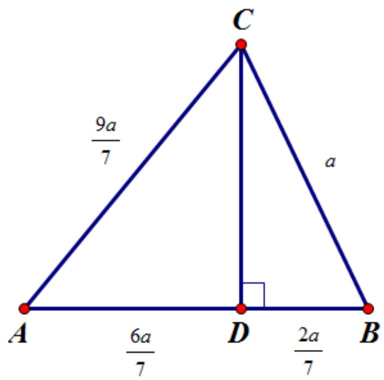
Solution 3. If a number is divisible by 3, then the sum of its digits is divisible by 3. The digits must be equivalent to one of the following (modulo 3): 0, 0, 0; 1, 1, 1; 2, 2, 2; or 0, 1, 2. Note that there are 3 possible nonzero digits congruent to each of 0, 1, and 2 (modulo 3). Therefore the number of ways to choose the three digits (without regard to their order) is

$$3 \cdot \binom{3}{3} + \binom{3}{1} \cdot \binom{3}{1} \cdot \binom{3}{1} = 3 \cdot 1 + 3 \cdot 3 \cdot 3 = 30.$$

Having chosen the three digits, consider the number of ways the three digits can be ordered. There are $3! = 6$ ways to order the three digits, and this gives a total of $30 \cdot 6 = \mathbf{180}$ three-digit numbers.

Problem 4. In $\triangle ABC$, $\cos(A) = \dfrac{2}{3}$ and $\cos(B) = \dfrac{2}{7}$. Given that the perimeter of $\triangle ABC$ is 24, compute the area of $\triangle ABC$.

Solution 4. Let D be the foot of the altitude from C. By the Law of Sines, $\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\sqrt{5}/3}{3\sqrt{5}/7} = \frac{7}{9}$. Thus $b = \frac{9a}{7}$, $AD = b \cos A = \frac{6a}{7}$, and $BD = a \cos B = \frac{2a}{7}$. Because the perimeter of $\triangle ABC$ is 24, solve $a + \frac{9a}{7} + \frac{6a}{7} + \frac{2a}{7} = 24$ to obtain $a = 7$, which implies $b = 9$ and $c = 8$. By Heron's Formula, $[ABC] = \sqrt{12 \cdot 5 \cdot 3 \cdot 4} = \mathbf{12\sqrt{5}}$.



Alternatively, drop altitude $\overline{CD}$ as before, and then apply the definition of cosine for $\angle CAD$ in $\triangle CAD$ and for $\angle CBD$ in $\triangle CBD$. Thus $AD = 2x$ and $AC = 3x$ for some real x and $DB = 2y$ and $CB = 7y$ for some real y. By the Pythagorean Theorem, $(CD)^2 = 9x^2 - 4x^2 = 5x^2$ and $(CD)^2 = 49y^2 - 4y^2 = 45y^2$. This implies $5x^2 = 45y^2$ so $x = 3y$. Substituting, $AD = 6y$ and $AC = 9y$. Because the perimeter of $\triangle ABC$ is 24, $9y+7y+6y+2y = 24 \rightarrow y = 1$, so $AB = 8$ and $CD = \sqrt{45y^2} = \sqrt{45} = 3\sqrt{5}$. Thus $[ABC] = \frac{1}{2}\cdot 8\cdot(3\sqrt{5}) = 12\sqrt{5}$.

Problem 5. Compute the product of all positive values of x that satisfy

$$\lfloor x+1\rfloor^{2x} - 19\lfloor x+1\rfloor^{x} + 48 = 0.$$

Solution 5. Introduce the substitution $y = \lfloor x+1\rfloor^{x}$. Then the given equation becomes $y^2 - 19y + 48 = 0$, and the left-hand side of the equation in y factors as $(y-3)(y-16) = 0$. Therefore $\lfloor x+1\rfloor^{x} = 3$ or $\lfloor x+1\rfloor^{x} = 16$.

Case 1: $y = \lfloor x+1\rfloor^{x} = 3$. Note that if $0 \leq x \leq 1$, then $\lfloor x+1\rfloor^{x} \leq 2$ and if $x \geq 2$, then $\lfloor x+1\rfloor^{x} \geq 9$. Therefore $1 < x < 2$, and hence $\lfloor x+1\rfloor = 2$. The equation $2^x = 3$ has the solution $x = \log_2 3$, and indeed, $1 < \log_2 3 < 2$, which checks.

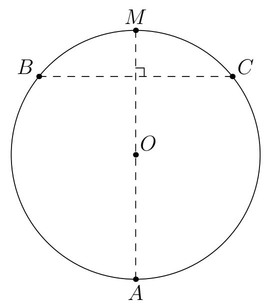
Case 2: $y = \lfloor x+1\rfloor^{x} = 16$. Note that if $0 \leq x \leq 2$, then $\lfloor x+1\rfloor^{x} \leq 9$ and if $x \geq 3$, then $\lfloor x+1\rfloor^{x} \geq 64$. Therefore $2 < x < 3$, and hence $\lfloor x+1\rfloor = 3$. The equation $3^x = 16$ has the solution $x = \log_3 16$, and indeed, $2 < \log_3 16 < 3$, which checks.

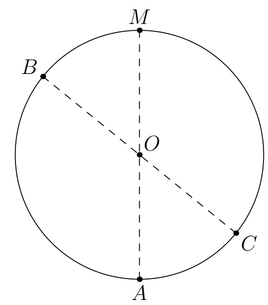
Use the change of base rule for logarithms to obtain the simplified product of the solutions:

$$(\log_2 3)(\log_3 16) = \frac{\log 3}{\log 2}\cdot\frac{\log 16}{\log 3} = \frac{\log 16}{\log 2} = \log_2 16 = \mathbf{4}.$$

Problem 6. Triangle ABC is inscribed in circle ω. The line containing the median from A meets ω again at M, the line containing the angle bisector of $\angle B$ meets ω again at R, and the line containing the altitude from C meets ω again at L. Given that quadrilateral $ARML$ is a rectangle, compute the degree measure of $\angle BAC$.

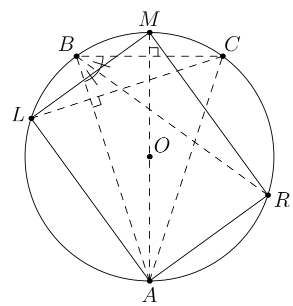
Solution 6. Because $ARML$ is a rectangle inscribed in a circle, its diagonal $\overline{AM}$ is a diameter of ω. The point B is on the circle and in one half-plane bounded by $\overleftrightarrow{AM}$. Because $\overline{AM}$ passes through the midpoint of side $\overline{BC}$, the point C is on the circle and in the opposite half-plane bounded by $\overleftrightarrow{AM}$ as B. Also, C must be the same distance from $\overline{AM}$ as B. Therefore either C is the reflection of B across line $\overline{AM}$ or C is the reflection of B across O, the center of ω, as shown below.





For $ARML$ to be a rectangle, it is sufficient that $\overline{LR}$ be a diameter of ω or, equivalently, that $\text{m}\angle LMR = 90°$. So consider the two cases for C, above, and determine if in either case it is possible to have $\text{m}\angle LMR = 90°$.

In the first case, $\triangle ABC$ is isosceles with $AB = AC$. Let $\beta = \text{m}\angle ABC = \text{m}\angle ACB$. Then $\text{m}\angle AMR = \text{m}\angle ABR = \frac{\beta}{2}$ and $\text{m}\angle AML = \text{m}\angle ACL = 180° - 90° - \text{m}\angle BAC = 2\beta - 90°$. Setting $\frac{\beta}{2} + 2\beta - 90° = 90°$ gives $\beta = 72°$. Then $\text{m}\angle BAC = 180° - 2\beta = 36°$.



In the second case, $\overline{BC}$ is a diameter of ω, so $\text{m}\angle BAC = 90°$. This means that L is coincident with A, which leads to a degenerate result. Thus the only possible value for $\text{m}\angle BAC$ is $\mathbf{36°}$.

Alternate Solution: Place the triangle in the complex plane so that ω coincides with the unit circle, and $a = 1$. (Here, a lowercase letter represents the complex coordinates of the point denoted by the corresponding uppercase letter.) Then $m = -1$ because diagonal $\overline{AM}$ of rectangle $ARML$ is a diameter of ω. Because the midpoint of $\overline{BC}$ has coordinates $\frac{b+c}{2}$ and because it lies on $\overline{AM}$ (which coincides with the real axis), it follows that $b + c \in \mathbb{R} \to b + c = \frac{1}{b} + \frac{1}{c} \to bc(b + c) = b + c$, and thus either $bc = 1$ or $b + c = 0$. The latter cannot be true, as it implies that $\angle BAC$ is a right angle, and thus $\ell = a$, $r = m = -1$, and then $c = a$, which results in a degenerate triangle. Hence $bc = 1$. Now, using some identities about the unit circle in the complex plane (which are straightforward to prove), conclude that $r^2 = ac$ (due to arc bisection) and $c\ell = -ab$ (from perpendicular chords). Expressing everything in terms of r:

$$c = \frac{r^2}{a} = r^2, \quad b = \frac{1}{c} = r^{-2}, \quad \text{and} \quad \ell = -\frac{ab}{c} = -r^{-4}.$$

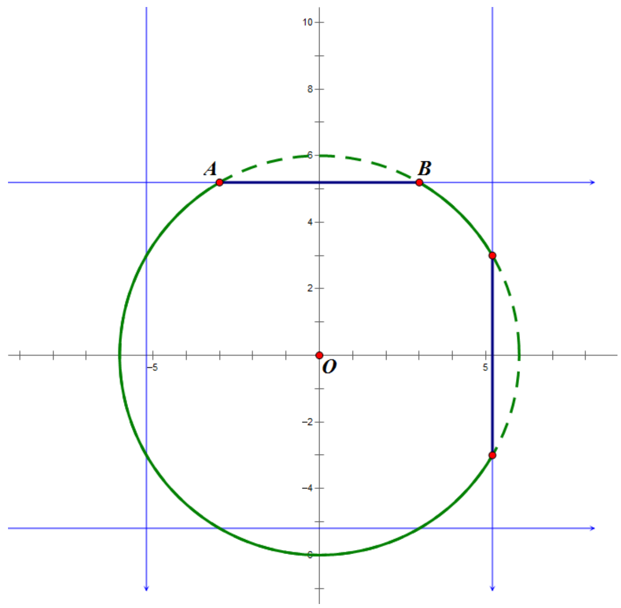
Because $ARML$ is a rectangle, its *other* diagonal, $\overline{RL}$, must also be a diameter of ω. Then $r = -\ell = r^{-4} \to r^5 = 1$, and so r is a fifth root of unity. It is impossible for r to be 1, as this would cause triangle ABC to be degenerate. Also, by replacing r with $\overline{r}$, note that the entire diagram is reflected across the real axis, and the measure of $\angle BAC$ does not change. Thus there are only two cases to consider: $r = e^{\frac{2\pi i}{5}}$ and $r = e^{\frac{4\pi i}{5}}$. Suppose that $r = e^{\frac{4\pi i}{5}}$. Then $c = e^{\frac{8\pi i}{5}}$ and $b = e^{\frac{2\pi i}{5}}$. But then B and L would lie on the same side of $\overline{AC}$, and $\overline{BL}$ would be the *external* angle bisector of $\angle B$. Thus $r = e^{\frac{2\pi i}{5}}$, which implies $b = e^{\frac{6\pi i}{5}}$ and $c = e^{\frac{4\pi i}{5}}$. Then minor arc BC measures $\frac{2\pi}{5}$ radians, which is $72°$, and so the degree measure of $\angle BAC$ is half of that, or $\mathbf{36°}$.

Problem 7. Let S be the set of points (x, y) whose coordinates satisfy $x^2 + y^2 \leq 36$ and $(\max\{x, y\})^2 \leq 27$. Compute the perimeter of S.

Solution 7. Let C be the region defined by $x^2 + y^2 \le 36$, and let R be the region defined by $(\max\{x, y\})^2 \le 27$, so $S = C \cap R$. If $(x, y) \in R$, then neither x nor y can be greater than $3\sqrt{3}$, and at least one of the coordinates must be between $-3\sqrt{3}$ and $3\sqrt{3}$. Thus if

$$R_1 = \left\{(x, y) \,:\, x \le 3\sqrt{3},\ |y| \le 3\sqrt{3}\right\} \text{ and}$$
$$R_2 = \left\{(x, y) \,:\, |x| \le 3\sqrt{3},\ y \le 3\sqrt{3}\right\},$$

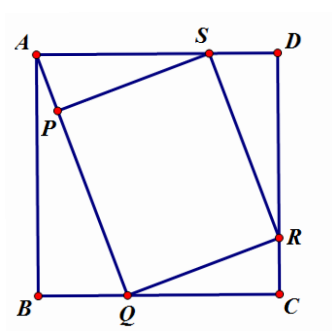
then $R = R_1 \cup R_2$. Consider the diagram below.



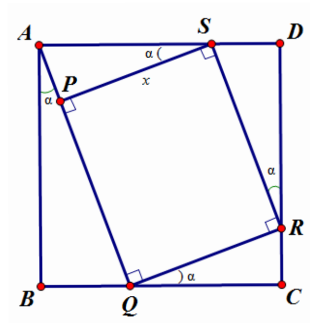
Most of circle C lies inside the central square $R_1 \cap R_2$, except for the segment cut off by the chord from $A(-3, 3\sqrt{3})$ to $B(3, 3\sqrt{3})$, and three other identical segments, none of which overlap. The segment on the left side of C lies inside R_1, and the bottom segment lies inside R_2. Thus S consists of the circle C with two such segments removed. If O is the origin, then $\triangle ABO$ is equilateral with side length 6, so the arc of the circle cut off by chord $\overline{AB}$ measures 60°. Thus removing each segment replaces a 60° arc of the circle with a chord of length 6, so the total perimeter of S, the region remaining, is

$$2\pi \cdot 6 - 2 \cdot \frac{1}{6}(2\pi \cdot 6) + 2 \cdot 6 = \boldsymbol{8\pi + 12}.$$

Problem 8. Rectangle $PQRS$ is drawn inside square $ABCD$, as shown. Given that $[APS] = 20$ and $[CQR] = 18$, compute $[ABCD]$.

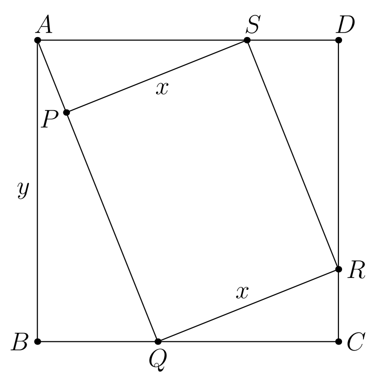


Solution 8. Notice first that triangles ABQ, QCR, SPA, and RDS are all similar. Let $PS = x$ and $m\angle ASP = \alpha$, as shown below.



Then $[APS] = \frac{PS \cdot AP}{2} = \frac{x \cdot x \tan\alpha}{2} = \frac{x^2 \tan\alpha}{2}$ and $[CQR] = \frac{CQ \cdot CR}{2} = \frac{x\cos\alpha \cdot x\cos\alpha\tan\alpha}{2} = \frac{x^2\tan\alpha}{2} \cdot \cos^2\alpha$. Hence $\frac{18}{20} = \cos^2\alpha \to \cos\alpha = \frac{3}{\sqrt{10}}$. Thus $\tan\alpha = \frac{1}{3}$. Now let $AB = y$. Then $BQ = \frac{y}{3}$, so $CQ = \frac{2y}{3}$ and $CR = \frac{2y}{9}$. Thus $[CQR] = \frac{2y^2}{27} = 18$, and so $[ABCD] = y^2 = \mathbf{243}$.

Alternate Solution: Because $[APS] = 20$, $[RCQ] = 18$, and $\triangle APS \sim \triangle RCQ$, conclude that $\dfrac{AS^2}{RQ^2} = \dfrac{20}{18}$, so $AS = \dfrac{\sqrt{10}}{3} \cdot RQ$. Let $RQ = x$. Then $RQ = PS = x$, thus $AS = \dfrac{\sqrt{10}}{3}x$, giving $AP^2 = \left(\dfrac{\sqrt{10}}{3}x\right)^2 - x^2 = \dfrac{x^2}{9}$, so $AP = \dfrac{x}{3}$. This implies $[APS] = \dfrac{1}{2} \cdot \dfrac{x}{3} \cdot x = 20$, so $x = 2\sqrt{30}$. Then $QC = \dfrac{3}{\sqrt{10}} \cdot 2\sqrt{30} = 6\sqrt{3}$. Now let $AB = y$. Because $\triangle ABQ \sim \triangle SPA$, it follows that $BQ = \dfrac{y}{3}$. From $y = BQ + QC$, conclude that $y = \dfrac{y}{3} + 6\sqrt{3} \to y = 9\sqrt{3}$. Thus the area of the square is $(9\sqrt{3})^2 = \mathbf{243}$.



Problem 9. Compute the least positive four-digit integer N for which N and $N+2018$ contain a total of 8 distinct digits.

Solution 9. Let $N = \underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ and $N + 2018 = \underline{E}\,\underline{F}\,\underline{G}\,\underline{H}$, where A, B, C, D, E, F, G, H represent distinct digits. To minimize N, first consider $A = 1$ and $B \neq 9$. Then $3018 < N + 2018 < 3918$, so $E = 3$. Because $B \neq F$, there must be a carry from the tens place to the hundreds place, and it follows that $C = 8$ or $C = 9$, and $B + 1 = F$. The least possible value for B is now 4, so assume $B = 4$ and $F = 5$. If there is no carry from the ones place to the tens place, then the only possibility for the ones digits is $D = 0$ and $H = 8$, but this requires $C = 9$ and $G = 0$ to achieve the carry into the hundreds place, which is a contradiction. Therefore there is a carry from the ones to the tens place. Because $G \neq 1$, it follows that $C = 8$ and $G = 0$. Finally, because $D + 8$

is the two-digit number $\underline{1}\,\underline{H}$, it must be that $D = H + 2$, and because $H \neq 0, 1, 3, 4, 5$ and $D \neq 4, 8$, the only possible values remaining are $D = 9$ and $H = 7$. Thus the least possible value of N is **1489**.

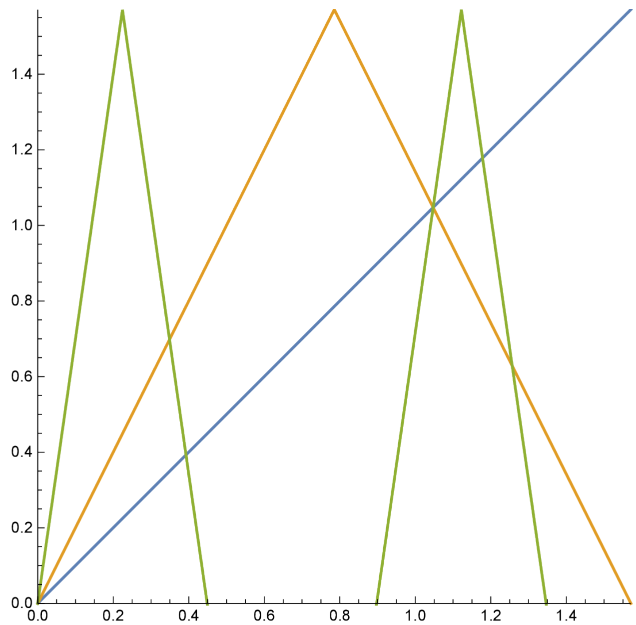
Problem 10. Compute the least positive value of t such that

$$\operatorname{Arcsin}(\sin(\alpha)),\ \operatorname{Arcsin}(\sin(2\alpha)),\ \operatorname{Arcsin}(\sin(7\alpha)),\ \operatorname{Arcsin}(\sin(t\alpha))$$

is a geometric progression for some α with $0 < \alpha < \frac{\pi}{2}$.

Solution 10. Let the ratio of consecutive terms in the sequence be r. If there exists an α for which the first three terms form a geometric progression with ratio r, the least t for which $\operatorname{Arcsin}\sin(t\alpha)$ continues the sequence will be $t = r^3$. This solution will focus on finding the least viable ratio r.

Note that as $2\alpha < \pi$, the first two terms of the progression are both positive, so the ratio r must also be positive. The graphs of $y = \operatorname{Arcsin}(\sin 2x)$ and $y = \operatorname{Arcsin}(\sin 7x)$ are both piecewise linear, the latter of which is positive only for $(0, \frac{\pi}{7}) \cup (\frac{2\pi}{7}, \frac{3\pi}{7})$, as seen below.



Thus consider only values of α in $(0, \frac{\pi}{7}) \cup (\frac{2\pi}{7}, \frac{3\pi}{7})$, because all terms in the progression must be positive. The function $\operatorname{Arcsin}(\sin 7x)$ is piecewise composed of four line segments in that domain. So, search along these segments to see if there exists an α for which a geometric progression results.

Along the first segment, no α exists, because the ratios between the first three terms are $\frac{2\alpha}{\alpha} = 2 \neq \frac{7\alpha}{2\alpha}$.

Along the second segment, $\operatorname{Arcsin}(\sin 7\alpha) = \pi - 7\alpha$, and the ratio equality becomes $\frac{2\alpha}{\alpha} = \frac{\pi - 7\alpha}{2\alpha} \to \alpha = \frac{\pi}{11}$. In this case, $r = 2$, and because $r^3\alpha = \frac{8\pi}{11} > \frac{\pi}{2}$, the geometric progression gets too large to be within the range of the Arcsin function.

Along the third segment, the three functions are equal to $x, \pi - 2x, 7x - 2\pi$, so the ratio equality is $\frac{\pi - 2\alpha}{\alpha} = \frac{7\alpha - 2\pi}{\pi - 2\alpha} \to \alpha = \frac{\pi}{3}$. In this case, $r = 1$ and the least t is $r^3 = 1$.

Finally, along the fourth segment, the three functions are equal to $x, \pi - 2x, 3\pi - 7x$, and the ratio equality is $\frac{\pi - 2\alpha}{\alpha} = \frac{3\pi - 7\alpha}{\pi - 2\alpha} \to \alpha = \frac{(7 \pm \sqrt{5})\pi}{22}$. One can check that the smaller value of α is extraneous while the larger, $\alpha = \frac{(7+\sqrt{5})\pi}{22}$, is not, and so $r = \frac{\pi - 2\alpha}{\alpha} = \frac{\pi}{\alpha} - 2 = \frac{3 - \sqrt{5}}{2}$, which gives $r^3 = 9 - 4\sqrt{5}$, which is positive because $9 - 4\sqrt{5} = \sqrt{81} - \sqrt{80} > 0$. This is the least value of t among all cases, so $t = \mathbf{9 - 4\sqrt{5}}$.

9 Relay Problems

Relay 1-1. The number $N = 327763$ can be expressed as

$$N = 51^3 + 58^3 = 30^3 + 67^3.$$

Compute the least prime factor of N.

Relay 1-2. Let $T = TNYWR$. Rectangle $ABCD$ has area $T + 5$. Point M is the midpoint of $\overline{AB}$ and point N is a trisection point on $\overline{CD}$. The segment $\overline{MN}$ divides rectangle $ABCD$ into two trapezoids. Compute the area of the larger of these two trapezoids.

Relay 1-3. Let $T = TNYWR$. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = \sqrt[4]{2}, a_2 = \sqrt{2}$, and for $n \geq 3$, $a_n = a_{n-1}a_{n-2}$. Compute the least value of k such that a_k is an integer multiple of $2^{\lfloor T \rfloor}$.

Relay 2-1. Given that A and B are integers such that $3A + 2B = 218$ and $|A - B|$ is as small as possible, compute $A + B$.

Relay 2-2. Let $T = TNYWR$. Compute the number of lattice points in the interior of the triangle with vertices $(0, T)$, $(T, 0)$, and $(0, 0)$.

Relay 2-3. Let $T = TNYWR$. The integers a and b are chosen from the set $\{1, 2, 3, \ldots, T\}$, with replacement. Compute the number of ordered pairs (a, b) for which

$$a + b = 4036.$$

10 Relay Answers

Answer 1-1. 31

Answer 1-2. 21

Answer 1-3. 11

Answer 2-1. 87

Answer 2-2. 3655

Answer 2-3. 3275

11 Relay Solutions

Relay 1-1. The number $N = 327763$ can be expressed as

$$N = 51^3 + 58^3 = 30^3 + 67^3.$$

Compute the least prime factor of N.

Solution 1-1. Note that the factorization $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ implies that $a + b$ is a divisor of $a^3 + b^3$. Thus $51 + 58 = 109$ and $30 + 67 = 97$ are each factors of N. From long division, $N = 31 \cdot 97 \cdot 109$, and each of the three factors in the product is prime, hence the answer is **31**.

Alternate Solution: Note that $N = 327 \cdot 1000 + 763$ and that $327 = 3 \cdot 109$ and $763 = 7 \cdot 109$. Thus $N = 109(3 \cdot 1000 + 7)$. As in the first solution, conclude that 97 must be a factor of 3007 and, using long division, the other factor is 31.

Note: Numbers that can be represented as the sum of two positive cubes in two different ways are examples of *taxicab numbers*. The least taxicab number is 1729 ($1729 = 10^3 + 9^3 = 12^3 + 1^3$). The origin of the name "taxicab number" comes from an anecdote involving a visit between mathematicians G. H. Hardy and S. Ramanujan. Hardy related that the number of the taxi he took to visit Ramanujan was 1729, and he remarked that the number seemed to be rather dull. Ramanujan countered by stating the fact discussed above.

Relay 1-2. Let $T = TNYWR$. Rectangle $ABCD$ has area $T + 5$. Point M is the midpoint of $\overline{AB}$ and point N is a trisection point on $\overline{CD}$. The segment $\overline{MN}$ divides rectangle $ABCD$ into two trapezoids. Compute the area of the larger of these two trapezoids.

Solution 1-2. Let $AB = CD = 6x$ and $BC = AD = y$. The smaller trapezoid has bases of lengths $2x$ and $3x$, so its area is $\frac{5}{2}xy$. The larger trapezoid has bases of lengths $3x$ and $4x$, so its area is $\frac{7}{2}xy$. Because $6xy = T + 5$, the area of the larger trapezoid is $\frac{7}{12}(T + 5)$. With $T = 31$, the desired area is **21**.

Relay 1-3. Let $T = TNYWR$. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = \sqrt[4]{2}, a_2 = \sqrt{2}$, and for $n \geq 3$, $a_n = a_{n-1}a_{n-2}$. Compute the least value of k such that a_k is an integer multiple of $2^{\lfloor T \rfloor}$.

Solution 1-3. Make a table consisting of the first few values of the sequence $a_1, a_2, a_3, \ldots$.

n	1	2	3	4	5	6
a_n	$2^{1/4}$	$2^{2/4}$	$2^{3/4}$	$2^{5/4}$	$2^{8/4}$	$2^{13/4}$

Note that a_n is of the general form $2^{F_{n+1}/4}$, where F_n denotes the n^{th} Fibonacci number. The desired value of k has the property that F_{k+1} is a multiple of 4 and is the least such value satisfying $F_{k+1} \geq 4\lfloor T \rfloor$. With $T = 21$, the least Fibonacci number that is both a multiple of 4 and which is greater than or equal to $4 \cdot 21 = 84$ is $F_{12} = 144$, hence $k = \mathbf{11}$.

Relay 2-1. Given that A and B are integers such that $3A + 2B = 218$ and $|A - B|$ is as small as possible, compute $A + B$.

Solution 2-1. Note that $|A - B| \neq 0$ because with $A = B$, $3A + 2B$ is a multiple of 5, but 218 is not a multiple of 5. Suppose there exist integers A and B such that $|A - B| = 1$. If $B = A + 1$, then the given equation becomes $5A + 2 = 218$, but this does not result in an integral value for A. On the other hand, if $A = B + 1$, then the given equation becomes $5B + 3 = 218$, and this yields the solution $(A, B) = (44, 43)$, hence $A + B = \mathbf{87}$.

Relay 2-2. Let $T = TNYWR$. Compute the number of lattice points in the interior of the triangle with vertices $(0, T)$, $(T, 0)$, and $(0, 0)$.

Solution 2-2. The following solution assumes that T is a positive integer. (As an exercise, the reader might want to consider how the solution would change if T were a negative integer or if T were not an integer.) It is easy to verify that if $T = 1$ or $T = 2$, then the answer is 0. So suppose that $T \geq 3$. For a fixed value of N with $1 \leq N \leq T - 2$, the points $(1, N), (2, N), \ldots, (T - N - 1, N)$ all lie in the interior of the triangle. This is a total of $T - N - 1$ points. Summing over the possible values of N yields $(T - 2) + (T - 3) + \cdots + 1 = \frac{(T-2)(T-1)}{2}$. With $T = 87$, the answer is **3655**.

Alternatively, one can apply Pick's Theorem, which states that for a polygon whose vertices are all lattice points, $K = I + \frac{B}{2} - 1$, where K is the area of the polygon, I is the number of lattice points in its interior, and B is the number of lattice points on its boundary. For the given triangle, $K = \frac{1}{2}T^2$ and $B = (T + 1) + T + (T - 1) = 3T$. Hence $I = \frac{T^2 - 3T + 2}{2}$, as established above.

Relay 2-3. Let $T = TNYWR$. The integers a and b are chosen from the set $\{1, 2, 3, \ldots, T\}$, with replacement. Compute the number of ordered pairs (a, b) for which

$$a + b = 4036.$$

Solution 2-3. The answer is the number of ordered pairs of integers $(x, 4036 - x)$ for which x and $4036 - x$ are both in the set $\{1, 2, 3, \ldots, T\}$. If $T < \frac{4036}{2} = 2018$, then the answer is 0 because the greatest possible sum would be achieved with the ordered pair (T, T), but $2T < 4036$. If $2018 \leq T < 4036$, the ordered pairs are $(4036 - T, T)$, $(4037 - T, T - 1), \ldots, (T, 4036 - T)$; there are $T - (4036 - T) + 1 = 2T - 4035$ of these ordered pairs. Finally, if $T \geq 4036$, then the answer is 4035 because each of the ordered pairs $(1, 4035), (2, 4034), \ldots, (4035, 1)$ has the sum of its coordinates equal to 4036. Because $T = 3655$, the desired number of ordered pairs is therefore $2 \cdot 3655 - 4035 =$ **3275**.

12 Super Relay

1. Given that a, b, and c are positive integers such that $a + bc = 20$ and $a + b = 18$, compute the least possible value of abc.

2. Let $T = TNYWR$. In convex quadrilateral $LEOS$, $LE = EO$, $LS = SO$, and $\overline{ES}$ and $\overline{LO}$ intersect in point J. Given that $EJ = 24$, $JS = 60$, and $EO = T$, compute LS.

3. Let $T = TNYWR$. For each integer n, let $f(n)$ be the remainder when $n^2 - 1$ is divided by 8. Compute $f(T) + f(T + 1) + f(T + 2) + f(T + 3)$.

4. Let $T = TNYWR$. Compute the shortest distance between the lines $y = x + T$ and $y = x - T$.

5. Let $T = TNYWR$. Compute the value of x for which $\sqrt{x} + \sqrt{x + 80} = T$.

6. Let $T = TNYWR$. Jane rolls a fair T-sided die whose faces are numbered from 1 to T inclusive. The probability that she rolls a multiple of either 2 or 3 is the same as the probability that she rolls a multiple of either 4, 5, 7, p, q, or r, where p, q, and r are prime numbers and $7 < p < q < r$. Compute r.

7. Let $T = TNYWR$, and let $K = T + 3$. Compute $\log_4 2^1 + \log_4 2^2 + \log_4 2^3 + \cdots + \log_4 2^{K-1} + \log_4 2^K$.

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15. Compute $\left|2018^3 - 3 \cdot 2018^2 \cdot 2017 + 3 \cdot 2018 \cdot 2017^2 - 2017^3\right|$.

14. Let $T = TNYWR$. In $\triangle DES$, $\sin D = \dfrac{1}{T}$ and $\sin E = \dfrac{1}{20}$. Given that $DS = 18$, compute ES.

13. Let $T = TNYWR$. A group of $T + 60$ students compete at an ARML site. At that site, a total of 50 students got at least one of individual questions 9 and 10 correct, 48 students got question 9 correct, and an astonishing 22 students got question 10 correct! Compute the probability that a randomly selected student from this group got *both* questions 9 and 10 correct.

12. Let $T = TNYWR$, and let $K = \dfrac{1}{T}$. The complex numbers a and b are the solutions to the equation $x^2 + 20x - K = 0$, and the complex numbers c and d are the solutions to the equation $x^2 - 18x + K - 4 = 0$. Compute $a^2 + b^2 + c^2 + d^2$.

11. Let $T = TNYWR$, and let P be the greatest prime factor of T. Given that a sphere has a surface area of $(P + 3)\pi$, compute the radius of the sphere.

10. Let $T = TNYWR$. Given that $f(x) = -x^2 + Tx + c$ for some real number c and $f(20) = 18$, compute the maximum value of $f(x)$, where x ranges over the real numbers.

9. Let $T = TNYWR$. In the increasing arithmetic sequence $a_1, a_2, a_3, \ldots$, the common difference between consecutive terms is 4, and $a_1 = T$. Compute the value of n for which $a_n = 2018$.

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8. Let A be the number you will receive from position 7 and let B be the number you will receive from position 9. A circular track has a perimeter of A meters. Jasmine and Richard start running clockwise around the track at the same time and from the same starting position on the track. Jasmine runs at a constant speed and takes 16 seconds to run A meters. Richard also runs at a constant speed and takes 160 seconds to run B meters. When Jasmine and Richard first meet again at the starting point, Jasmine will have run m laps and Richard will have run n laps. Compute $20m + 18n$.

13 Super Relay Answers

1. 51

2. 75

3. 10

4. $10\sqrt{2}$

5. 18

6. 17

7. 105

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15. 1

14. 360

13. $\dfrac{1}{21}$

12. 732

11. 4

10. 342

9. 420

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8. 136

14 Super Relay Solutions

Problem 1. Given that a, b, and c are positive integers such that $a + bc = 20$ and $a + b = 18$, compute the least possible value of abc.

Solution 1. Subtract the second equation from the first equation and factor to obtain $b(c-1) = 2$. Hence either $(b, c) = (2, 2)$ or $(b, c) = (1, 3)$. In the former case, $a = 18 - 2 = 16$, and in the latter case, $a = 18 - 1 = 17$. The value of abc is thus minimized when $a = 17$, $b = 1$, and $c = 3$, and $abc = \mathbf{51}$.

Problem 2. Let $T = TNYWR$. In convex quadrilateral $LEOS$, $LE = EO$, $LS = SO$, and $\overline{ES}$ and $\overline{LO}$ intersect in point J. Given that $EJ = 24$, $JS = 60$, and $EO = T$, compute LS.

Solution 2. Note that quadrilateral $LEOS$ is a kite, hence $\overline{ES} \perp \overline{LO}$. By the Pythagorean Theorem, $JO^2 = T^2 - 24^2$ and $LS^2 = OS^2 = JO^2 + JS^2 = T^2 - 24^2 + 60^2 = T^2 + 3024$, hence $LS = \sqrt{T^2 + 3024}$. With $T = 51$, $LS = \mathbf{75}$. (Note: The calculation can be simplified by noting that once a value of T is received, $\triangle EJO$ is similar to an $8-15-17$ triangle, and $\triangle OJS$ is similar to a $3-4-5$ triangle.)

Problem 3. Let $T = TNYWR$. For each integer n, let $f(n)$ be the remainder when $n^2 - 1$ is divided by 8. Compute $f(T) + f(T+1) + f(T+2) + f(T+3)$.

Solution 3. Note that exactly two elements of $\{T, T+1, T+2, T+3\}$ are even, and the other two elements of the set are odd. Similarly, exactly two elements of $\{T^2 - 1, (T+1)^2 - 1, (T+2)^2 - 1, (T+3)^2 - 1\}$ are even, and the other two elements of the set are odd. Without loss of generality, assume that $T = 4k$, for some integer k. Then $f(T) = 16k^2 - 1 \equiv 7 \pmod 8$, $f(T+1) = (4k)(4k+2) \equiv 0 \pmod 8$, $f(T+2) = 16k^2 + 8k + 3 \equiv 3 \pmod 8$, and $f(T+3) = (4k+2)(4k+4) \equiv 0 \pmod 8$. Hence the answer is $7 + 0 + 3 + 0 = \mathbf{10}$ (independent of T).

Problem 4. Let $T = TNYWR$. Compute the shortest distance between the lines $y = x + T$ and $y = x - T$.

Solution 4. Let $A = (0, T)$, $B = (0, -T)$, and let C be the foot of the perpendicular from A to the line $y = x - T$. Note that the two lines are parallel and, because their slopes are 1, they make an angle of inclination of 45° with the x-axis. Hence $m\angle ABC = m\angle BAC = 45^\circ$. Because $AB = |2T|$, the distance between the two lines, AC, is equal to $|T\sqrt{2}|$. With $T = 10$, the answer is therefore $\mathbf{10\sqrt{2}}$.

Problem 5. Let $T = TNYWR$. Compute the value of x for which $\sqrt{x} + \sqrt{x + 80} = T$.

Solution 5. Rewrite the given equation as $\sqrt{x+80} = T - \sqrt{x}$ and square each side to obtain $x + 80 = T^2 - 2T\sqrt{x} + x$. Thus $2T\sqrt{x} = T^2 - 80$, hence $x = \left(\frac{T^2 - 80}{2T}\right)^2$. With $T = 10\sqrt{2}$, the value of x is $\mathbf{18}$, which checks.

Problem 6. Let $T = TNYWR$. Jane rolls a fair T-sided die whose faces are numbered from 1 to T inclusive. The probability that she rolls a multiple of either 2 or 3 is the same as the probability that she rolls a multiple of either 4, 5, 7, p, q, or r, where p, q, and r are prime numbers and $7 < p < q < r$. Compute r.

Solution 6. Equivalently, the number of outcomes that are multiples of 2 or 3 should equal the number of outcomes that are multiples of 4, 5, 7, p, q, or r. There are $\lfloor T/2 \rfloor$ multiples of 2 and $\lfloor T/3 \rfloor$ multiples of 3. Each of these counts multiples of 2 *and* 3, so by the Inclusion-Exclusion Principle, the number of multiples of 2 or 3 is $\lfloor T/2 \rfloor + \lfloor T/3 \rfloor - \lfloor T/6 \rfloor$. Similarly, the number of outcomes that are multiples of 4, 5, 7, p, q, or r is bounded above by $\lfloor T/4 \rfloor + \lfloor T/5 \rfloor + \lfloor T/7 \rfloor + \lfloor T/p \rfloor + \lfloor T/q \rfloor + \lfloor T/r \rfloor$. Depending on the value of T, this count may need to be adjusted in the event that an outcome is a multiple of at least two of the numbers 4, 5, 7, p, q, and r. However, if $T < 20$, this will not happen. With $T = 18$, the number of multiples of 2 or 3 is $9 + 6 - 3 = 12$, and the number of multiples of 4, 5, 7, p, q, or r is $4 + 3 + 2 + 1 + 1 + 1 = 12$. The only possible triple (p, q, r) is $(11, 13, 17)$, hence $r = \mathbf{17}$.

Problem 7. Let $T = TNYWR$, and let $K = T + 3$. Compute $\log_4 2^1 + \log_4 2^2 + \log_4 2^3 + \cdots + \log_4 2^{K-1} + \log_4 2^K$.

Solution 7. The given expression is equal to

$$\begin{aligned}\log_4\left(2^1 \cdot 2^2 \cdot \ldots \cdot 2^{K-1} \cdot 2^K\right) &= \log_4\left(2^{1+2+\cdots+(K-1)+K}\right)\\ &= \log_4\left(\left(4^{1/2}\right)^{K(K+1)/2}\right)\\ &= K(K+1)/4\\ &= (T+3)(T+4)/4.\end{aligned}$$

With $T = 17$, the answer is **105**.

Problem 15. Compute $\left|2018^3 - 3 \cdot 2018^2 \cdot 2017 + 3 \cdot 2018 \cdot 2017^2 - 2017^3\right|$.

Solution 15. Let $a = 2018$ and $b = 2017$. Then the given expression equals $\left|a^3 - 3a^2b + 3ab^2 - b^3\right| = \left|(a-b)^3\right|$. Because $a - b = 1$, the answer is therefore $\left|1^3\right| = \mathbf{1}$.

Problem 14. Let $T = TNYWR$. In $\triangle DES$, $\sin D = \frac{1}{T}$ and $\sin E = \frac{1}{20}$. Given that $DS = 18$, compute ES.

Solution 14. By the Law of Sines, $\frac{ES}{\sin D} = \frac{DS}{\sin E}$, so $ES = DS \cdot \frac{\sin D}{\sin E} = 18 \cdot \frac{20}{T} = \frac{360}{T}$. Because $T = 1$, the answer is **360** (and $\angle D$ is a right angle).

Problem 13. Let $T = TNYWR$. A group of $T + 60$ students compete at an ARML site. At that site, a total of 50 students got at least one of individual questions 9 and 10 correct, 48 students got question 9 correct, and an astonishing 22 students got question 10 correct! Compute the probability that a randomly selected student from this group got *both* questions 9 and 10 correct.

Solution 13. Let N be the number of students who got both questions 9 and 10 correct. By the Inclusion-Exclusion Principle, it follows that $50 = 48 + 22 - N$, so $N = 20$. Thus the desired probability is $\frac{20}{T+60}$. With $T = 360$, this gives an answer of $\mathbf{\frac{1}{21}}$.

Problem 12. Let $T = TNYWR$, and let $K = \frac{1}{T}$. The complex numbers a and b are the solutions to the equation $x^2 + 20x - K = 0$, and the complex numbers c and d are the solutions to the equation $x^2 - 18x + K - 4 = 0$. Compute $a^2 + b^2 + c^2 + d^2$.

Solution 12. By Vieta's Formulas, $a+b = -20$, $ab = -K$, $c+d = 18$, $cd = K-4$. Note also that $a^2+b^2+c^2+d^2 = (a+b)^2+(c+d)^2-2ab-2cd = (-20)^2+18^2-2(-K)-2(K-4) = 400+324+2K-2K+8 = \mathbf{732}$ (independent of K and T).

Problem 11. Let $T = TNYWR$, and let P be the greatest prime factor of T. Given that a sphere has a surface area of $(P+3)\pi$, compute the radius of the sphere.

Solution 11. Let r be the radius of the sphere. Then $4\pi r^2 = (P+3)\pi$, so $r = \dfrac{\sqrt{P+3}}{2}$. With $T = 732 = 2 \cdot 3^2 \cdot 61$, $P = 61$, and the answer is $\frac{\sqrt{64}}{2} = \mathbf{4}$.

Problem 10. Let $T = TNYWR$. Given that $f(x) = -x^2 + Tx + c$ for some real number c and $f(20) = 18$, compute the maximum value of $f(x)$, where x ranges over the real numbers.

Solution 10. Plug in $x = 20$ to obtain $18 = f(20) = -400 + 20T + c$, hence $c = 418 - 20T$. Complete the square in the quadratic polynomial to obtain $f(x) = -\left(x - \frac{T}{2}\right)^2 + \frac{T^2}{4} + 418 - 20T$. When x ranges over the real numbers, f will be maximized when $x = \frac{T}{2}$ because $-\left(x - \frac{T}{2}\right)^2$ is negative except when $x = \frac{T}{2}$. Thus the maximum value of f is $\frac{T^2}{4} + 418 - 20T$. With $T = 4$, this yields the answer of $\mathbf{342}$.

Problem 9. Let $T = TNYWR$. In the increasing arithmetic sequence $a_1, a_2, a_3, \ldots$, the common difference between consecutive terms is 4, and $a_1 = T$. Compute the value of n for which $a_n = 2018$.

Solution 9. The desired value of n satisfies the equation $T + 4(n-1) = 2018$, thus $n = \dfrac{2022 - T}{4}$. With $T = 342$, this yields $n = \mathbf{420}$.

Problem 8. Let A be the number you will receive from position 7 and let B be the number you will receive from position 9. A circular track has a perimeter of A meters. Jasmine and Richard start running clockwise around the track at the same time and from the same starting position on the track. Jasmine runs at a constant speed and takes 16 seconds to run A meters. Richard also runs at a constant speed and takes 160 seconds to run B meters. When Jasmine and Richard first meet again at the starting point, Jasmine will have run m laps and Richard will have run n laps. Compute $20m + 18n$.

Solution 8. Note that Jasmine takes 16 seconds to run a lap around the track, whereas Richard takes $\frac{160}{B} \cdot A$ seconds to run a lap around the track. Let $\ell = \operatorname{lcm}(16, 160A/B)$. Then Jasmine and Richard will first meet again at the starting point after ℓ seconds have elapsed. At that time, Jasmine will have run a total of $\frac{\ell}{16}$ laps and Richard will have run a total of $\frac{\ell}{160A/B}$ laps. With $A = 105$ and $B = 420$, it follows that $\ell = 80$ and therefore $m = \frac{80}{16} = 5$ and $n = \frac{80}{40} = 2$. Therefore $20m + 18n = 100 + 36 = \mathbf{136}$.

15 Tiebreaker Problems

Problem 1. The increasing infinite arithmetic sequence of integers $x_1, x_2, x_3, \ldots$ contains the terms $17!$ and $18!$. Compute the greatest integer X for which $X!$ must also appear in the sequence.

Problem 2. Let $\mathcal{S}_1 = \{1, 2, 3, 4\}$, $\mathcal{S}_2 = \{3, 4, 5, 6\}$, and $\mathcal{S}_3 = \{6, 7, 8\}$. Compute the number of sets $\mathcal{H}$ that satisfy both of the following properties:

- $\mathcal{H} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$;
- each of $\mathcal{H} \cap \mathcal{S}_1$, $\mathcal{H} \cap \mathcal{S}_2$, and $\mathcal{H} \cap \mathcal{S}_3$ is non-empty.

Problem 3. Tom writes down the integers from 1 to 100, inclusive, on a chalkboard and then erases every number that contains a prime digit. Compute the sum of the *digits* that remain on the chalkboard.

16 Tiebreaker Answers

Answer 1. 32

Answer 2. 201

Answer 3. 337

17 Tiebreaker Solutions

Problem 1. The increasing infinite arithmetic sequence of integers $x_1, x_2, x_3, \ldots$ contains the terms $17!$ and $18!$. Compute the greatest integer X for which $X!$ must also appear in the sequence.

Solution 1. Because the sequence contains $17!$ and $18!$, its common difference must divide $M = 18! - 17! = 17 \cdot 17!$. Note that the set of integers appearing in all such sequences forms an arithmetic sequence with first term $17!$ and common difference M. The problem is thus equivalent to finding the greatest integer X such that $X! = 18! + kM$ for some integer k. Divide both sides of the previous equation by $17!$, to obtain $X(X-1)(X-2) \cdot \ldots \cdot 19 \cdot 18 = 18 + 17k$, or in other words,

$$X(X-1)(X-2) \cdot \ldots \cdot 19 \cdot 18 \equiv 18 \pmod{17}. \tag{$*$}$$

Note that modulo 17, the left-hand side of $(*)$ is equal to $(X-17)!$, hence the problem reduces to finding the greatest solution to $(X-17)! \equiv 1 \pmod{17}$. There is no such solution with $X > 33$, because then $(X-17)!$ is divisible by 17. Also, if $X = 33$, then by Wilson's Theorem, $16! \equiv -1 \pmod{17}$ because 17 is prime. However, $16! = 15! \cdot 16 \equiv 15! \cdot (-1) \equiv -1 \pmod{17}$, hence $15! \equiv 1 \pmod{17}$. Thus the desired value of X is **32**.

Problem 2. Let $\mathcal{S}_1 = \{1, 2, 3, 4\}$, $\mathcal{S}_2 = \{3, 4, 5, 6\}$, and $\mathcal{S}_3 = \{6, 7, 8\}$. Compute the number of sets $\mathcal{H}$ that satisfy both of the following properties:

- $\mathcal{H} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$;
- each of $\mathcal{H} \cap \mathcal{S}_1$, $\mathcal{H} \cap \mathcal{S}_2$, and $\mathcal{H} \cap \mathcal{S}_3$ is non-empty.

Solution 2. There are a total of 2^8 subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Proceed indirectly and consider counting the sets $\mathcal{H}$ such that at least one of $\mathcal{H} \cap \mathcal{S}_1$, $\mathcal{H} \cap \mathcal{S}_2$, and $\mathcal{H} \cap \mathcal{S}_3$ is empty. Let N_a be the number of subsets $\mathcal{H}$ of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ in which $\mathcal{H} \cap \mathcal{S}_a$ is empty. Define $N_{a,b}$ and $N_{a,b,c}$ similarly. Then, by the Inclusion-Exclusion Principle, the answer is

$$2^8 - \big(N_1 + N_2 + N_3 - (N_{1,2} + N_{1,3} + N_{2,3}) + N_{1,2,3}\big).$$

Because $|\mathcal{S}_1| = 4$, it follows that $N_1 = 2^{8-4} = 16$. Similarly, $N_2 = 16$ and $N_3 = 32$. There are 6 elements in $\mathcal{S}_1 \cup \mathcal{S}_2$, thus $N_{1,2} = 2^{8-6} = 4$. Similarly, $N_{1,3} = 2$ and $N_{2,3} = 4$. Finally, $N_{1,2,3} = 1$ because $\emptyset$ is the only subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that has an empty intersection with each of $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. Thus the answer is $256 - \big(16 + 16 + 32 - (4 + 2 + 4) + 1\big) = \mathbf{201}$.

Alternate Solution: Define a set $\mathcal{H}$ with the desired properties to be a *hitting set*. Count the number of hitting sets $\mathcal{H}$ according to the possible sizes of the set $\mathcal{H}$.

$\lvert\mathcal{H}\rvert$	**Number of hitting sets $\mathcal{H}$**
$0, 1$	0: A set $\mathcal{H}$ with $\lvert\mathcal{H}\rvert \le 1$ cannot have a non-empty intersection with each of the three sets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ because there is no element common to all three of these sets.
2	8: Note that $\lvert\mathcal{H} \cap \mathcal{S}_3\rvert = 1$. If $6 \in \mathcal{H}$, then because $\mathcal{H} \cap \mathcal{S}_1 \neq \emptyset$, There are 4 possible elements of $\mathcal{S}_1$ that could belong to $\mathcal{H}$. If $6 \notin \mathcal{H}$, then there are 2 choices of the element of $\mathcal{S}_3$ that could belong to $\mathcal{H}$ (7 or 8) and there are two choices of the element of $\mathcal{S}_1 \cap \mathcal{S}_2$ that could belong to $\mathcal{H}$ (3 or 4). Thus if $6 \notin \mathcal{H}$, then there are $2 \cdot 2 = 4$ possible hitting sets $\mathcal{H}$, for a total of $4 + 4 = 8$ hitting sets of size 2.
3	$\binom{8}{3} - 18 = 38$: Note that every subset $\mathcal{H}$ of $\{1,2,3,4,5,6,7,8\}$ with $\lvert\mathcal{H}\rvert = 3$ will have a non-empty intersection with at least two of the sets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. There are 4 sets $\mathcal{H}$, each of which has an empty intersection with $\mathcal{S}_1$; these are the $\binom{4}{3}$ subsets of $\{5,6,7,8\}$ of size 3. There are 4 sets $\mathcal{H}$, each of which has an empty intersection with $\mathcal{S}_2$; these are the $\binom{4}{3}$ subsets of $\{1,2,7,8\}$ of size 3. Finally, there are 10 sets $\mathcal{H}$, each of which has an empty intersection with $\mathcal{S}_3$; these are the $\binom{5}{3}$ subsets of $\{1,2,3,4,5\}$ of size 3. Thus there are $\binom{8}{3} - (4 + 4 + 10) = 38$ hitting sets of size 3.
4	$\binom{8}{4} - 7 = 63$: Note that every subset $\mathcal{H}$ of $\{1,2,3,4,5,6,7,8\}$ with $\lvert\mathcal{H}\rvert = 4$ will have a non-empty intersection with at least two of the sets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. There is 1 set $\mathcal{H}$ that has an empty intersection with $\mathcal{S}_1$: $\{5,6,7,8\}$. There is 1 set $\mathcal{H}$ that has an empty intersection with $\mathcal{S}_2$: $\{1,2,7,8\}$. Finally, there are 5 sets $\mathcal{H}$, each of which has an empty intersection with $\mathcal{S}_3$: $\{1,2,3,4\}$, $\{1,2,3,5\}$, $\{1,2,4,5\}$, $\{1,3,4,5\}$, and $\{2,3,4,5\}$. Thus there are $\binom{8}{4} - (1 + 1 + 5) = 63$ hitting sets of size 4.
5	$\binom{8}{5} - 1 = 55$: Any set $\mathcal{H}$ with $\lvert\mathcal{H}\rvert = 5$ will intersect each of $\mathcal{S}_1$ and $\mathcal{S}_2$ in at least one element. The only set of size 5 that does not intersect $\mathcal{S}_3$ is $\{1,2,3,4,5\}$, hence there are $\binom{8}{5} - 1 = 55$ hitting sets of size 5.
6	$\binom{8}{6} = 28$: Because each of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ has at least 3 elements, a 6-element hitting set will have a non-empty intersection with each of $\mathcal{S}_1, \mathcal{S}_2$, and $\mathcal{S}_3$.
7	$\binom{8}{7} = 8$: Because each of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ has at least 3 elements, a 7-element hitting set will have a non-empty intersection with each of $\mathcal{S}_1, \mathcal{S}_2$, and $\mathcal{S}_3$.
8	1: The only possible set is $\{1,2,3,4,5,6,7,8\}$, which clearly works.

From the above analysis, the total number of hitting sets is $0 + 0 + 8 + 38 + 63 + 55 + 28 + 8 + 1 = \mathbf{201}$.

Note: As in the alternate solution, a set $\mathcal{H}$ with the property that it has a non-empty intersection with each set in a collection of sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ is known as a *hitting set*. Intuitively, this name makes sense because $\mathcal{H}$ "hits" (i.e., intersects) each set $\mathcal{S}_i$ such that $\lvert\mathcal{H} \cap \mathcal{S}_i\rvert \ge 1$. The following general problem is believed to be computationally hard in the sense that there are no known efficient algorithms for answering it.

> *Given a collection of n sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ and a positive integer k, does there exist a hitting set $\mathcal{H}$ with $\lvert\mathcal{H}\rvert \le k$?*

Problem 3. Tom writes down the integers from 1 to 100, inclusive, on a chalkboard and then erases every number that contains a prime digit. Compute the sum of the *digits* that remain on the chalkboard.

Solution 3. Every one- or two-digit integer can be represented as $\underline{T}\,\underline{U}$ where $0 \le T, U \le 9$ (and for one-digit integers, $T = 0$). The prime digits are 2, 3, 5, and 7, so Tom will erase the 40 numbers of the form $\underline{2}\,\underline{U}$, $\underline{3}\,\underline{U}$, $\underline{5}\,\underline{U}$, and $\underline{7}\,\underline{U}$. After that, consider a fixed value of T, not equal to 2, 3, 5, or 7. Tom will erase the numbers

$\underline{T\,2}$, $\underline{T\,3}$, $\underline{T\,5}$, and $\underline{T\,7}$. Among the six numbers remaining with tens digit T, the sum of their units digits is $0+1+4+6+8+9=28$, and the sum of their tens digits is $6T$. There are six such values of T, so the sum of all the digits of the remaining one- and two-digit integers is $6 \cdot 28 + 6(0+1+4+6+8+9) = 12 \cdot 28 = 336$. This sum includes the "number" 00, but the zeros do not affect the sum. Finally, because the number 100 remains, the answer is $336 + 1 = \mathbf{337}$.