# 2018 ARML Local Solutions
## Team Round Solutions

T-1 A sphere with surface area $2118\pi$ is circumscribed around a cube and a smaller sphere is inscribed in the cube. Compute the surface area of the smaller sphere.

**Solution:** If the cube has side length $D$, the sphere inscribed inside the cube has diameter $D$, and the circumscribed sphere has diameter $\sqrt{3}D$. The surface area of a cube of side length $D$ is $\pi D^2$, so the inscribed sphere has surface area $\frac{1}{3}$ of the circumscribed one, so the answer is $\boxed{706\pi}$.

T-2 $REBS$ is a square of side length 3. If $SU = UO = OR = RT = TX = XE = ED = DI = IB = BM = MA = AS = 1$, compute the sum of the areas of the distinct rectangles whose vertices consist of four of the points $A, M, B, I, D, E, X, T, R, O, U, S$.

**Solution:** The other eight points given are the trisection points for the edges of $REBS$. There are rectangles of five different dimensions: $3 \times 1$ (there are six), $3 \times 2$ (there are four), $3 \times 3$ (there's one), $\sqrt{2} \times \sqrt{8}$ (there are two), and $\sqrt{5} \times \sqrt{5}$ (there are two). The sum of the areas is $6 \times 3 + 4 \times 6 + 1 \times 9 + 2 \times 4 + 2 \times 5 = \boxed{69}$.

T-3 Compute the value of $x$ for which $\log_8(\log_4 x) = \log_{64}(\log_2 x)$.

**Solution:** Let $x = 2^u$, then $\log_2 x = u$ and $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{u}{2}$. Then $\log_8(u/2) = \log_{64}(u) = \frac{\log_8(u)}{\log_8 64} = \frac{\log_8(u)}{2} \to \log_8(u) - \log_8(2) = \frac{\log_8(u)}{2} \to \frac{\log_8(u)}{2} = \frac{1}{3} \to \log_8(u) = \frac{2}{3} \to u = 4 \to x = \boxed{16}$.

T-4 For non-negative integers $A$ and $B$, with $A \geq B$, let $A \star B$ denote the number formed by the concatenation of $(A+B)$, $(A-B)$, and $(A \times B)$. For example, $7 \star 5 = 12235$. Compute the number of ordered pairs $(A, B)$, $1 \leq B \leq A \leq 9$, such that $A \star B$ contains at most three distinct digits (to clarify, the example contains four distinct digits).

**Solution:** The resulting number will be at most five digits for single-digit values of $A$ and $B$. The cases that satisfy the conditions of the problem are:

- Three digits long: The sum, difference, and product all being single digits: this happens for $B = 1$, $1 \leq A \leq 8$, $B = 2$, $2 \leq A \leq 4$, and $B = 3$, $A = 3$.
- Four digits: The sum must be a single digit and will be distinct from the difference. Therefore, the product must share a digit with either the sum or the difference. This only occurs when $A = 4$ and $B = 3$ (7112).
- Five digits: The tens digit of the sum is always one. If the difference is also one, then either the sum is 11 ($6 \star 5 = 11130$), or the sum shares a digit with the product ($8 \star 7 = 15156$ and $9 \star 8 = 17172$). If the difference is not one, then either the sum is 11 and the product contains a one ($9 \star 2 = 11718$) or the difference and product

contain a common digit ($8\star 2 = 10616$), or the sum and product contain two common digits ($9 \star 9 = 18081$), or the sum, difference, and product contain a common digit ($9 \star 5 = 14445$) .

Counting the cases, the answer is $\boxed{20}$.

T-5 Compute the greatest integer $n$ for which $(6!)^n$ is a factor of $60!$.

**Solution:** $(6!)^n = 2^{4n}3^{2n}5^n$. Therefore, we determine the largest power of 2, 3, and 5 that divides $60!$. In general, the largest power of $p$ that divides $k!$ is $\sum_{j=1}^{\infty}\lfloor \frac{k}{p^j} \rfloor$. When $p = 2$, this sum is $30 + 15 + 7 + 3 + 1 = 56$, when $p = 3$, this sum is $20 + 6 + 2 = 28$, and when $p = 5$, the sum is $12 + 2 = 14$. Accordingly, $\boxed{14}$ is largest power of $6!$ that divides $60!$ evenly.

T-6 The ARML Local staff have ranked each of the 10 individual problems in order of difficulty from 1 to 10. They wish to order the problems such that the problem with difficulty $i$ is before the problem with difficulty $i + 2$ for all $1 \le i \le 8$. Compute the number of orderings of the problems that satisfy this condition.

**Solution:** Any acceptable ordering of the problems must have the odd numbered problems and even numbered problems in order of increasing difficulty. Therefore, it is merely a question of how many ways two sequences of 5 numbers can be interleaved. There are $C(10, 5) = \boxed{252}$ ways to pick the positions of the questions of odd difficulty, and then a unique ordering of the odd difficulty questions in those positions and the even difficulty questions in the positions left over.

T-7 Let $ABC$ be a triangle with $AB = 5$, $BC = 12$, and an obtuse angle at $B$. It is given that there exists a point $P$ in plane $ABC$ for which $\angle PBC = \angle PCB = \angle PAB = \gamma$, where $\gamma$ is an acute angle satisfying $\tan\gamma = \frac{3}{4}$. Compute $\sin\angle ABP$.

**Solution:** Note that by Law of Sines on $\triangle ABP$ and $\triangle CBP$,

$$\frac{AB}{\sin\angle APB} = \frac{PB}{\sin\gamma} = \frac{BC}{\sin\angle BPC},$$

from which

$$\sin\angle APB = \frac{AB}{BC}\cdot\sin\angle BPC = \frac{5}{12}\cdot\sin 2\gamma = \frac{5}{12}\cdot\frac{24}{25} = \frac{2}{5}.$$

It follows that

$$\begin{aligned}\sin\angle ABP &= \sin(\angle BAP + \angle APB) \\ &= \sin\angle BAP\cos\angle APB + \sin\angle APB\cos\angle APB \\ &= \frac{2}{5}\cdot\frac{4}{5} + \frac{3}{5}\cdot\frac{\sqrt{21}}{5} = \boxed{\frac{8+3\sqrt{21}}{25}}.\end{aligned}$$

T-8 Suppose $a$, $b$, and $c$ are real numbers such that $a, b, c$ and $a, b+1, c+2$ are both geometric sequences in their respective orders. Compute the smallest possible value of $(a + b + c)^2$.

**Solution:** Note that the conditions imply $b^2 = ac$ and $(b+1)^2 = a(c+2)$. Subtracting yields $2b+1 = 2a$, so $b + \frac{1}{2} = a$. Now recall that from the first equation $c = \frac{b^2}{a}$, so

$$a+b+c = a + \left(\frac{1}{2}+a\right) + \frac{(\frac{1}{2}+a)^2}{a} = 3a + \frac{1}{4a} - \frac{3}{2}.$$

If $a$ is positive, then this expression is at least $2\sqrt{\frac{4}{3}} - \frac{3}{2} = \sqrt{3} - \frac{3}{2}$ by the AM-GM inequality; in a similar vein, $a$ is negative this expression is at most $-\sqrt{3} - \frac{3}{2}$. From both of these cases we conclude that the smallest possible value of $(a+b+c)^2$ is $(\sqrt{3}-\frac{3}{2})^2 = \boxed{\frac{21}{4} - 3\sqrt{3}}$.

T-9 Circles $\omega_1$ and $\omega_2$ have radii 5 and 12 respectively and have centers distance 13 apart. Let $A$ and $B$ denote the intersection points of $\omega_1$ and $\omega_2$. A line $\ell$ passing through $A$ intersects $\omega_1$ again at $X$ and $\omega_2$ again at $Y$ such that $\angle ABY = 2\angle ABX$. Compute $XY$.

**Solution:** Denote by $O_1$ and $O_2$ the centers of $\omega_1$ and $\omega_2$ respectively. Then $O_1A^2 + O_2A^2 = O_1O_2^2$ from the lengths given in the problem statement, so $O_1A \perp O_2A$. Now set $\angle XAO_1 = \alpha$ and $\angle YAO_2 = \beta$; then $\alpha + \beta = 90^\circ$. Furthermore, angle chasing reveals

$$\angle ABX = \tfrac{1}{2}\angle AO_1X = 90^\circ - \alpha \qquad \text{and} \qquad \angle ABY = \tfrac{1}{2}\angle AO_2Y = 90^\circ - \beta,$$

and so $90^\circ - \beta = 2(90^\circ - \alpha) \Rightarrow 2\alpha - \beta = 90^\circ$. Solving this system of equations yields $\alpha = 60^\circ$ and $\beta = 30^\circ$, so

$$XY = XA + AY = 2\cdot 5\cos 60^\circ + 2\cdot 12\cos 30^\circ = \boxed{5 + 12\sqrt{3}}.$$

T-10 Compute the sum of all possible values of $a+b$, where $a$ and $b$ are integers such that $a > b$ and $a^2 - b^2 = 2016$.

**Solution:** The condition in the problem to be satisfied is equivelent to $(a-b)(a+b) = 2016$. Note that $a - b > 0$, so $a + b > 0$ also. For $d = a + b$, note that $d \mid 2016$, and furthermore $d$ and $2016/d$ must have the same parity. It follows that $d = 2^a 3^b 7^c$ for $a \in \{1,2,3,4\}$, $b \in \{0,1,2\}$, and $c \in \{0,1\}$; the requested sum is

$$(2+2^2+2^3+2^4)(1+3+3^2)(1+7) = 30\cdot 13\cdot 8 = \boxed{3120}.$$

T-11 If $u$ and $v$ are complex numbers such that $u+v = 9$ and $u^3 + v^3 = 81$, compute $u^2 + v^2$.

**Solution:** $81 = (u+v)(u^2 - uv + v^2) \to (u^2 - uv + v^2) = 9$. $u + v = 9 \to u^2 + 2uv + v^2 = 81$. Combining the two equations to eliminate the $u^2$ and $v^2$ terms gives $3uv = 72 \to uv = 24$, so $u^2 + v^2 = 9 + uv = \boxed{33}$.

T-12 On the planet ARMLia, there are three species of sentient creatures: Trickles, who count in base 3, Quadbos, who count in base 4, and Quinters, who count in base 5. One thing all three species agree on is the magic of the positive integer Zelf, which ends in the digits "11" when represented in all three bases. Compute the least possible value (in base 10) of Zelf.

**Solution:** Let $z$ be the least possible value of Zelf. This becomes a Chinese Remainder Theorem problem, where $z \equiv (b+1) \mod b^2$ for each base $b$. Looking first at Trickles and Quadbos, $z \equiv 4 \mod 9$ and $z \equiv 5 \mod 16$.

Using the Euclidean algorithm, we establish that $4 \times 16 - 7 \times 9 = 1$, so $z \equiv 4 \times 4 \times 16 - 5 \times 7 \times 9 \equiv -59 \mod 144 \equiv 85 \mod 144$. Additionally, $z \equiv 6 \mod 25$, and $4 \times 144 - 23 \times 25 = 1$, so $z \equiv 4 \times 144 \times 6 - 23 \times 25 \times 85 \equiv 3456 - 48875 = -45419 \equiv \boxed{1381} \mod 3600$.

T-13 Compute the number of ordered pairs of integers $(a, b, c)$ with $a \geq b \geq c$ such that $a$, $b$, and $c$ are the side lengths of a non-degenerate triangle with perimeter 218.

**Solution:** To satisfy the triangle inequality, $a > b + c$. The longest side $a$ can take values between 108 $(108 - 108 - 2, 108 - 107 - 3, \ldots 108 - 55 - 55)$ and 73 $(73 - 73 - 72)$. For a fixed value of $a$, $b$ can vary between $\lceil \frac{218-a}{2} \rceil$ and $a$, inclusive, while $c$ is fixed given the choice of $a$ and $b$ $(c = 218 - a - b)$.

$$\begin{aligned}
\sum_{a=73}^{108} a - \lceil \frac{218-a}{2} \rceil + 1 &= \sum_{a=73}^{108} a - \sum_{a=73}^{108} \lceil \frac{218-a}{2} \rceil + \sum_{a=73}^{108} 1 \\
&= 181 \times 18 - (73 + 72 + 72 + \cdots + 56 + 56 + 55) + 36 \\
&= 3258 - 2304 + 36 = \boxed{990}
\end{aligned}$$

T-14 Compute the sum of the real roots of $f(x) = x^6 + 3x^4 - 2018x^3 + 3x^2 + 1$.

**Solution:** If $f(x) = 0$, then $2018x^3 = x^6 + 3x^4 + 3x^2 + 1 = (x^2 + 1)^3$. Dividing both sides by $x^3$ gives $2018 = (\frac{x^2+1}{x})^3 = (x + \frac{1}{x})^3$. By symmetry, this means that if $x$ is a root, so is $\frac{1}{x}$, so the sum of the real roots is $(x + \frac{1}{x}) = \boxed{\sqrt[3]{2018}}$.

T-15 Place in the $10 \times 10$ grid of cells below one $4 \times 4$ square, two $3 \times 3$ squares, three $2 \times 2$ squares, and four $1 \times 1$ squares such that:

- All square sides are parallel to boundaries of the grid.
- No squares overlap nor share a common boundary, not even a corner point.
- The total number of $1 \times 1$ cells covered by squares in each row and column is equal to the number to the right or below the row or column, respectively. The numbered or lettered cells may not be covered by tiles.

Enter on your answer sheet the letter coordinates (given in the first row and column) of the four cells containing the $1 \times 1$ squares, for example, if the four corners of the grid contained the $1 \times 1$ tiles, you would enter $AK$, $JK$, $AT$, and $JT$.

| | A | B | C | D | E | F | G | H | I | J | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| K | | | | | | | | | | | 3 |
| L | | | | | | | | | | | 2 |
| M | | | | | | | | | | | 7 |
| N | | | | | | | | | | | 4 |
| O | | | | | | | | | | | 7 |
| P | | | | | | | | | | | 7 |
| Q | | | | | | | | | | | 3 |
| R | | | | | | | | | | | 5 |
| S | | | | | | | | | | | 7 |
| T | | | | | | | | | | | 5 |
| | 3 | 6 | 4 | 8 | 7 | 3 | 2 | 3 | 7 | 7 | |

**Solution:** The only consecutive rows and columns that contain 4 or more filled squares are M to P and B to E, respectively, so the $4 \times 4$ must lie there. Additionally, no other filled squares are in A-F $\times$ L-Q, and no other filled squares in column C or row N. Because column D contains eight filled squares, DK must be filled in with a $1 \times 1$ and D-F $\times$ R-T must be filled in with one of the $3 \times 3$ squares. This eliminates EK and FK from being filled in, as the conditions on columns E and F are now satisfied, as well as G $\times$ Q-T, as those squares are adjacent to the $3 \times 3$. The only way that rows O and P can contain 7 filled in squares is if the other $3 \times 3$ square is at H-J $\times$ O-Q, which leaves the only way to satisfy the condition on row R to place a $2 \times 2$ square at A-B $\times$ R-S, and then $2 \times 2$ squares at I-J $\times$ S-T and I-J $\times$ L-M. This leaves the positions of the other three $1 \times 1$ squares at AK, GK, and GM. The answer is $\boxed{\text{AK, DK, GK, and GM}}$. The fully filled in solution is below.

| | A | B | C | D | E | F | G | H | I | J | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| K | X | | | X | | | X | | | | 3 |
| L | | | | | | | | | X | X | 2 |
| M | | X | X | X | X | | X | | X | X | 7 |
| N | | X | X | X | X | | | | | | 4 |
| O | | X | X | X | X | | | X | X | X | 7 |
| P | | X | X | X | X | | | X | X | X | 7 |
| Q | | | | | | | | X | X | X | 3 |
| R | X | X | | X | X | X | | | | | 5 |
| S | X | X | | X | X | X | | | X | X | 7 |
| T | | | | X | X | X | | | X | X | 5 |
| | 3 | 6 | 4 | 8 | 7 | 3 | 2 | 3 | 7 | 7 | |

# Individual Round Solutions

I-1 The number $N = 2^{12} - 1$ is the product of 5 (not necessarily distinct) prime numbers $p_1$, $p_2$, $p_3$, $p_4$, and $p_5$. Compute $p_1 + p_2 + p_3 + p_4 + p_5$.

**Solution:** $2^{12} - 1 = (2^6 - 1)(2^6 + 1) = (2^3 - 1)(2^3 + 1)(2^6 + 1) = 7 \times 9 \times 65 = 3 \times 3 \times 5 \times 7 \times 13$. The sum of the primes is $\boxed{31}$.

I-2 If $a_0, a_1, a_2, \ldots$ is an arithmetic sequence and $18a_{20} + 20 = 20a_{18} + 18$, compute $a_0$.

**Solution:** Let $d$ be the common difference of the sequence, then $a_{20} = a_{18} + 2d \rightarrow 18(a_{18}+2d)+20 = 20a_{18}+18 \rightarrow 2a_{18} = 36d+2$ or $a_{18} = 18d+1$. Since $a_{18} = a_0+18d$, the answer is $\boxed{1}$. An alternate solution is that since the sequence isn't specified, you could assume it is a constant sequence, in which case $18a_0 + 20 = 20a_0 + 18$, and the answer follows.

I-3 If $x$ and $y$ are real numbers such that $2x + y = 4$ and $5x + 3y = 9$, compute the value of $16x + 9y$.

**Solution:** Summing three times the third equation and two times the second, you get that $3(2x + y) + 2(5x + 3y) = 3 \times 4 + 2 \times 9 \rightarrow 16x + 9y = 12 + 18 = \boxed{30}$.

I-4 For each real number $x$, let $f(x) = \sin(\frac{\pi x}{3}) + \cos(\frac{\pi x}{2})$. Compute the least $p > 0$ such that $f(x + p) = f(x)$ for all real $x$.

**Solution:** The first summand $\sin(\frac{\pi x}{3})$ has period 6, since $\sin(x)$ has period $2\pi$. Similarly, the second summand has period 4. The period of the sum has period equal to the least common multiple of the periods of the two summands, which is $\boxed{12}$.

I-5 Compute the number of triples of consecutive positive integers less than 50 whose product is both a multiple of 20 and 18.

**Solution:** The least common multiple of 20 and 18 is 180. One of the values in the triple must be a multiple of 5 (say $m$), and then if the greatest common divisor of $m$ and 180 is $d$, the product of the other two terms must be a multiple of $\frac{180}{d}$. This leads to the following triples that satisfy the condition:

– $m = 10$, $d = 18$, and the sequence 8,9,10
– $m = 20$, $d = 9$, and the sequence 18,19,20
– $m = 35$, $d = 36$, and the sequences 34,35,36 and 35,36,37
– $m = 45$, $d = 4$, and the sequences 43,44,45 and 44,45,46

Therefore, the answer is $\boxed{6}$.

I-6 Suppose that $x$ is a complex number that satisfies $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$. Compute the product $x^{20} \cdot x^{18} \cdot x^2 \cdot x^0 \cdot x^1 \cdot x^8$.

**Solution:** The left hand side of the equation is equal to $\frac{x^7-1}{x-1}$, so the values that satisfy the equation are the seventh roots of unity not equal to 1. $x^{20} \cdot x^{18} \cdot x^2 \cdot x^0 \cdot x^1 \cdot x^8 = x^{49}$, so for any seventh root of unity, its $49^{th}$ power will be $\boxed{1}$.

I-7 Let $S = \{1, 2, 3, 4, 5, 6\}$. Compute the number of invertible $2 \times 2$ matrices with entries that are distinct elements of $S$.

**Solution:** There are $6 \times 5 \times 4 \times 3 = 360$ $2 \times 2$ matrices with entries that are distinct elements of $S$. If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M$ is not invertible if and only if $ad = bc$. There are two sets of pairs of distinct elements of $S$ with equal products, $1 \times 6 = 2 \times 3$ and $2 \times 6 = 3 \times 4$. For each set of pairs, any of the four values can be equal to $a$, which then sets the corresponding value in $d$, and then the other pair can fill the remaining positions in two ways in $M$. Therefore, sixteen of the matrices are non-invertible, and the answer is $360 - 16 = \boxed{344}$.

I-8 Let $ABCD$ be a rectangle with $AB = 15$ and $BC = 19$. Points $E$ and $F$ are located inside $ABCD$ such that quadrilaterals $AECF$ and $BEDF$ are both parallelograms with areas 107 and 88 respectively. Compute the maximum value of $EF$.

**Solution:** Note that since $AECF$ is a parallelogram, $EF$ passes through the midpoint of $\overline{AC}$, which just so happens to be the center of the rectangle $ABCD$. Embed the figure into a coordinate system with the origin the center of $ABCD$, and WLOG set $A = (-\frac{19}{2}, -\frac{15}{2})$, $B = (-\frac{19}{2}, \frac{15}{2})$, $C = (\frac{19}{2}, \frac{15}{2})$, and $D = (\frac{19}{2}, -\frac{15}{2})$. Let $E = (x, y)$ (WLOG assume that $E$ is clockwise from $A$ in $AECF$), so that $F = (-x, -y)$. Using the Shoelace theorem on parallelogram $AECF$ gives:

$$\begin{aligned}107 = [AECF] &= \frac{|\frac{-19}{2}y + \frac{15}{2}x + \frac{-19}{2}y + \frac{15}{2}x - (\frac{-15}{2}x + \frac{19}{2}y + \frac{-15}{2}x + \frac{19}{2}y)|}{2} \\ &= |15x - 19y|.\end{aligned}$$

Applying similar reasoning to $BEDF$ yields the equation $|15x + 19y| = 88$.

Assuming WLOG that $y > 0$, there are two possible cases: either $15x - 19y = -107$ and $15x + 19y = -88$ (which has the solution $x = \frac{-13}{2}$, $y = \frac{1}{2}$) or $15x - 19y = -107$ and $15x + 19y = 88$ (which has the solution $x = \frac{-19}{30}$, $y = \frac{195}{38}$). $EF = 2\sqrt{x^2 + y^2}$, and the greater value occurs in the first case, where $EF = 2\sqrt{(\frac{-13}{2})^2 + (\frac{1}{2})^2} = \boxed{\sqrt{170}}$.

I-9 Compute the number of positive integers $N$ between 1 and 100 inclusive have the property that there exist distinct divisors $a$ and $b$ of $N$ such that $a + b$ is also a divisor of $N$.

**Solution:** First note that by scaling downward we may assume $\gcd(a, b) = 1$. This implies $\gcd(a, a + b) = 1$ and $\gcd(b, a + b) = 1$, so $N$ in fact is divisible by $ab(a + b)$. We now compute numbers of that form which are $\leq 100$ and remove all the ones which are multiples of any others. This leaves 6, 20, 56, and 70 as the only four possibilities (most of the other numbers are multiples of six). It thus suffices to compute the number of integers between 1 and 100 which are multiples of at least one of these four numbers. There are $\lfloor \frac{100}{6} \rfloor = 16$ multiples of six between 1 and 100. Then there are four more multiples of 20 (20, 40, 80, 100). Including 56 and 70 gives $\boxed{22}$ total integers.

I-10 It is given that there exist real numbers $a_0, \ldots, a_{18}$ such that

$$\frac{\sin^2(10x)}{\sin^2(x)} = a_0 + a_1\cos(x) + a_2\cos(2x) + \cdots + a_{18}\cos(18x)$$

for all $x$ with $\sin x \neq 0$. Compute $a_0^2 + a_1^2 + \cdots + a_{18}^2$.

**Solution:** Recall that $\sin t = \frac{e^{it}-e^{-it}}{2}$ for all $t$. Thus

$$\frac{\sin(10x)}{\sin(x)} = \frac{e^{10ix} - e^{-10ix}}{e^{ix} - e^{-ix}} = e^{9ix} + e^{7ix} + \cdots + e^{-7ix} + e^{9ix}$$

for all $x$. This means that

$$\begin{aligned}\frac{\sin^2(10x)}{\sin^2(x)} &= \left(e^{9ix} + e^{7ix} + \cdots + e^{-7ix} + e^{-9ix}\right)^2 \\ &= \left(e^{18ix} + e^{-18ix}\right) + 2\left(e^{16ix} + e^{-16ix}\right) + \cdots + 9\left(e^{2ix} + e^{-2ix}\right) + 10 \\ &= 2\cos(18x) + 4\cos(16x) + \cdots + 18\cos(2x) + 10.\end{aligned}$$

The requested answer is $10^2 + (2^2 + 4^2 + \cdots + 18^2) = \boxed{1240}$.

# Relay Round Solutions

R1-1 Compute the sum of the values of $x$ such that $x^{(x-4)^2} = x^{16}$.

**Solution:** There are three cases where the equality holds: either the exponents are equal ($(x-4)^2 = 16 \to x = 0$ or $8$), or the base $x = 0$ or $1$. The other possibility is $x = -1$ if the exponent is even, but $(-5)^2$ is odd. The answer is $\boxed{9}$.

R1-2 Let $T = TNYWR$. Compute the remainder when $\sum_{i=1}^{3}(x-4i)^{(4-i)}$ is divided by $x - T$.

**Solution:** The sum expands to $(x-4)^3 + (x-8)^2 + (x-12)$ which can be rewritten as $((x-T)+(T-4))^3 + ((x-T)+(T-8))^2 + ((x-T)+(T-12))$ which when divided by $x-T$ leaves $(T-4)^3 + (T-8)^2 + (T-12) = 5^3 + 1^2 - 3 = \boxed{123}$.

R2-1 Leah rolls a fair standard six-sided die three times. Compute the probability that the product of the three rolls is prime.

**Solution:** The product of the three rolls is prime if and only if two of the rolls are 1 and one of the rolls is 2, 3, or 5. There are nine sequences of three rolls that satisfy these conditions out of $6^3$, so the probability is $\frac{9}{216} = \boxed{\frac{1}{24}}$.

R2-2 Let $T = TNYWR$. Compute the greatest integer $N$ such that $8^{NT} < 4$.

**Solution:** Taking the logarithm (base-2) of both sides becomes $3NT < 2 \to N < \frac{2}{3T}$. As $T = \frac{1}{24}$, $N < 16$, which means the answer is $\boxed{15}$.

R2-3 Let $T = TNYWR$. Compute the sum of the perimeters of all distinct rectangles with integer side lengths and area $T$.

**Solution:** Let the side lengths of the rectangle be $x$ and $y$, and WLOG, $x \le y$. The area of the rectangle is $xy$ and the perimeter is $2x + 2y$. Since $T = 15$, there are two pairs of factors $(x, y)$ such that $x \le y$ and $xy = 15$: $(1, 15)$ and $(3, 5)$. The sum of the perimeters is $32 + 16 = \boxed{48}$.

R3-1 Compute the sum of all prime numbers $p$ such that $p^{2018} + p^{2019}$ is a perfect square.

**Solution:** Factoring out $p^{2018}$, we see that the value is a perfect square if and only if $1 + p = k^2$ for some integer $k$. $p = k^2 - 1 = (k-1)(k+1)$, so $p$ is prime only when $k = 2$, or $p = \boxed{3}$.

R3-2 Let $T = TNYWR$. Compute the number of integers in the domain of the function $f(x) = \sqrt{\log((2-x)(x+T))}$.

**Solution:** In order for $f(x)$ to be defined, $\log((2-x)(x+T) \ge 0 \to (2-x)(x+T) \ge 1 \to (x-2)(x+T) \le 1 \to x^2 + (T-2)x - (2T-1) \le 0$. As $T = 3$, $x^2 + x - 5 \le 0 \to$

$\frac{-1-\sqrt{21}}{2} \leq x \leq \frac{-1+\sqrt{21}}{2}$. The set of integers that satisfy these inequalities are $-2 \leq x \leq 1$, so the answer is $\boxed{4}$.

R3-3 Let $T = TNYWR$. Compute the number of ordered triplets $(a, b, c)$ such that $1 \leq a, b, c \leq T$ and $a + b + c$ is odd.

**Solution:** There are $T^3$ triplets in total. If $T$ is even, there are an equal number of triplets that have even and odd sums, so the answer is $\frac{4^3}{2} = \boxed{32}$.

R3-4 Let $T = TNYWR$. The lengths of the two perpendicular legs of a right triangle sum to $T$. A circle of radius $r$ is tangent to all three sides of the triangle, and a circle of radius $R$ passes through all three vertices of the triangle. Compute $r + R$.

**Solution:** The inradius of a right triangle with legs $a$ and $b$ is $\frac{a+b-\sqrt{a^2+b^2}}{2}$. The radius the circumcircle is $\frac{\sqrt{a^2+b^2}}{2}$, their sum is $\frac{a+b}{2} = \frac{T}{2} = \boxed{16}$.

R3-5 Let $T = TNYWR$. Compute the number of integers between 100 and 999 inclusive whose digits sum to $T$ that are *not* multiples of 5.

**Solution:** If $k = \underline{X}\,\underline{Y}\,\underline{Z}$. Let $A(n)$ be the number of pairs of digits $\underline{X}\,\underline{Y}$ that sum to $n$. Note that $A(n) = n$ if $T \leq 9$ and $19 - n$ for $10 \leq T \leq 18$, and 0 otherwise. Ignoring the multiples of 5, the total number of three digit numbers is $A(T-1) + A(T-2) + A(T-3) + A(T-4) + A(T-6) + A(T-7) + A(T-8) + A(T-9)$. As $T = 16$, the sum is $4 + 5 + 6 + 7 + 9 + 9 + 8 + 7 = \boxed{55}$.

R3-6 Let $T = TNYWR$. The number of subsets of size $N$ of the letters in the word TEAMWORK that do not contain all of the vowels (A, E, and O) is $T$. Compute $N$.

**Solution:** The total number of subsets of size $N$ is $\binom{8}{N}$. Of them, $\binom{N}{N-3}$ contain all of the vowels, so $T = \binom{8}{N} - \binom{N}{N-3}$. Since $T = 55$, $N = \boxed{3}$.

# Tiebreaker Solution

TB For all nonnegative integers $c$, let $f(c)$ denote the unique real number $x$ satisfying the equation
$$\frac{x^5 + x^4 - 48}{x^9 - 1} = c.$$
Compute $f(0)f(1)f(2)\cdots f(47)$.

**Solution:** Rewrite the original condition as $P_c(x) := cx^9 - x^5 - x^4 + (48 - c) = 0$. Now remark that
$$P_{48-c}(x) = (48 - c)x^9 - x^5 - x^4 + c = x^9 P_c(\tfrac{1}{x});$$
this means that the roots of $P_c(x)$ and $P_{48-c}(x)$ (which are nonzero since $c \neq 48$) are reciprocals of each other, so $f(c)f(48 - c) = 1$ for any $c$. Now just compute $f(0) = 2$ and $f(24) = -1$; this gives the final answer as $\boxed{-2}$.