2018 ARML Local Problems

Team Round (45 minutes)

T-1 A sphere with surface area 2118π is circumscribed around a cube and a smaller sphere is inscribed in the cube. Compute the surface area of the smaller sphere.

T-2 $REBS$ is a square of side length 3. If $SU = UO = OR = RT = TX = XE = ED = DI = IB = BM = MA = AS = 1$, compute the sum of the areas of the distinct rectangles whose vertices consist of four of the points $A, M, B, I, D, E, X, T, R, O, U, S$.

T-3 Compute the value of x for which $\log_8(\log_4 x) = \log_{64}(\log_2 x)$.

T-4 For non-negative integers A and B, with $A \geq B$, let $A \star B$ denote the number formed by the concatenation of $(A + B)$, $(A - B)$, and $(A \times B)$. For example, $7 \star 5 = 12235$. Compute the number of ordered pairs (A, B), $1 \leq B \leq A \leq 9$, such that $A \star B$ contains at most three distinct digits (to clarify, the example contains four distinct digits).

T-5 Compute the greatest integer n for which $(6!)^n$ is a factor of $60!$.

T-6 The ARML Local staff have ranked each of the 10 individual problems in order of difficulty from 1 to 10. They wish to order the problems such that the problem with difficulty i is before the problem with difficulty $i + 2$ for all $1 \leq i \leq 8$. Compute the number of orderings of the problems that satisfy this condition.

T-7 Let ABC be a triangle with $AB = 5$, $BC = 12$, and an obtuse angle at B. It is given that there exists a point P in plane ABC for which $\angle PBC = \angle PCB = \angle PAB = \gamma$, where γ is an acute angle satisfying $\tan\gamma = \frac{3}{4}$. Compute $\sin\angle ABP$.

T-8 Suppose a, b, and c are real numbers such that a, b, c and $a, b+1, c+2$ are both geometric sequences in their respective orders. Compute the smallest possible value of $(a + b + c)^2$.

T-9 Circles ω_1 and ω_2 have radii 5 and 12 respectively and have centers distance 13 apart. Let A and B denote the intersection points of ω_1 and ω_2. A line ℓ passing through A intersects ω_1 again at X and ω_2 again at Y such that $\angle ABY = 2\angle ABX$. Compute XY.

T-10 Compute the sum of all possible values of $a + b$, where a and b are integers such that $a > b$ and $a^2 - b^2 = 2016$.

T-11 If u and v are complex numbers such that $u + v = 9$ and $u^3 + v^3 = 81$, compute $u^2 + v^2$.

T-12 On the planet ARMLia, there are three species of sentient creatures: Trickles, who count in base 3, Quadbos, who count in base 4, and Quinters, who count in base 5. One thing all three species agree on is the magic of the positive integer Zelf, which ends in the digits "11" when represented in all three bases. Compute the least possible value (in base 10) of Zelf.

T-13 Compute the number of ordered pairs of integers (a, b, c) with $a \geq b \geq c$ such that a, b, and c are the side lengths of a non-degenerate triangle with perimeter 218.

T-14 Compute the sum of the real roots of $f(x) = x^6 + 3x^4 - 2018x^3 + 3x^2 + 1$.

T-15 Place in the 10×10 grid of cells below one 4×4 square, two 3×3 squares, three 2×2 squares, and four 1×1 squares such that:

- All square sides are parallel to boundaries of the grid.
- No squares overlap nor share a common boundary, not even a corner point.
- The total number of 1×1 cells covered by squares in each row and column is equal to the number to the right or below the row or column, respectively. The numbered or lettered cells may not be covered by tiles.

Enter on your answer sheet the letter coordinates (given in the first row and column) of the four cells containing the 1×1 squares, for example, if the four corners of the grid contained the 1×1 tiles, you would enter AK, JK, AT, and JT.

	A	B	C	D	E	F	G	H	I	J	
K											3
L											2
M											7
N											4
O											7
P											7
Q											3
R											5
S											7
T											5
	3	6	4	8	7	3	2	3	7	7	

Individual Round (10 minutes per pair)

I-1 The number $N = 2^{12} - 1$ is the product of 5 (not necessarily distinct) prime numbers p_1, p_2, p_3, p_4, and p_5. Compute $p_1 + p_2 + p_3 + p_4 + p_5$.

I-2 If $a_0, a_1, a_2, \ldots$ is an arithmetic sequence and $18a_{20} + 20 = 20a_{18} + 18$, compute a_0.

I-3 If x and y are real numbers such that $2x + y = 4$ and $5x + 3y = 9$, compute the value of $16x + 9y$.

I-4 For each real number x, let $f(x) = \sin(\frac{\pi x}{3}) + \cos(\frac{\pi x}{2})$. Compute the least $p > 0$ such that $f(x + p) = f(x)$ for all real x.

I-5 Compute the number of triples of consecutive positive integers less than 50 whose product is both a multiple of 20 and 18.

I-6 Suppose that x is a complex number that satisfies $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$. Compute the product $x^{20} \cdot x^{18} \cdot x^2 \cdot x^0 \cdot x^1 \cdot x^8$.

I-7 Let $S = \{1, 2, 3, 4, 5, 6\}$. Compute the number of invertible 2×2 matrices with entries that are distinct elements of S.

I-8 Let $ABCD$ be a rectangle with $AB = 15$ and $BC = 19$. Points E and F are located inside $ABCD$ such that quadrilaterals $AECF$ and $BEDF$ are both parallelograms with areas 107 and 88 respectively. Compute the maximum value of EF.

I-9 Compute the number of positive integers N between 1 and 100 inclusive have the property that there exist distinct divisors a and b of N such that $a + b$ is also a divisor of N.

I-10 It is given that there exist real numbers $a_0, \ldots, a_{18}$ such that

$$\frac{\sin^2(10x)}{\sin^2(x)} = a_0 + a_1 \cos(x) + a_2 \cos(2x) + \cdots + a_{18} \cos(18x)$$

for all x with $\sin x \neq 0$. Compute $a_0^2 + a_1^2 + \cdots + a_{18}^2$.

Relay Round (6 minutes, 8 minutes, 10 minutes)

R1-1 Compute the sum of the values of x such that $x^{(x-4)^2} = x^{16}$.

R1-2 Let $T = TNYWR$. Compute the remainder when $\sum_{i=1}^{3}(x-4i)^{(4-i)}$ is divided by $x - T$.

R2-1 Leah rolls a fair standard six-sided die three times. Compute the probability that the product of the three rolls is prime.

R2-2 Let $T = TNYWR$. Compute the greatest integer N such that $8^{NT} < 4$.

R2-3 Let $T = TNYWR$. Compute the sum of the perimeters of all distinct rectangles with integer side lengths and area T.

R3-1 Compute the sum of all prime numbers p such that $p^{2018} + p^{2019}$ is a perfect square.

R3-2 Let $T = TNYWR$. Compute the number of integers in the domain of the function $f(x) = \sqrt{\log((2-x)(x+T))}$.

R3-3 Let $T = TNYWR$. Compute the number of ordered triplets (a, b, c) such that $1 \leq a, b, c \leq T$ and $a + b + c$ is odd.

R3-4 Let $T = TNYWR$. The lengths of the two perpendicular legs of a right triangle sum to T. A circle of radius r is tangent to all three sides of the triangle, and a circle of radius R passes through all three vertices of the triangle. Compute $r + R$.

R3-5 Let $T = TNYWR$. Compute the number of integers between 100 and 999 inclusive whose digits sum to T that are *not* multiples of 5.

R3-6 Let $T = TNYWR$. The number of subsets of size N of the letters in the word TEAMWORK that do not contain all of the vowels (A, E, and O) is T. Compute N.

Tiebreaker (10 minutes)

TB For all nonnegative integers c, let $f(c)$ denote the unique real number x satisfying the equation

$$\frac{x^5 + x^4 - 48}{x^9 - 1} = c.$$

Compute $f(0)f(1)f(2)\cdots f(47)$.