Funny Factorials and Slick Sums—Solutions

Spring 2017 ARML Power Contest

Problem 1. We apply the definition of Δ to each function.
(a) $f(x+1) - f(x) = c - c = 0$.
(b) $f(x+1) - f(x) = a(x+1) + b - (ax+b) = a$.
(c) $f(x+1) - f(x) = (x+1)^3 - x^3 = x^3 + 3x^2 + 3x + 1 - x^3 = 3x^2 + 3x + 1$.
(d) $f(x+1) - f(x) = 2^{x+1} - 2^x = 2 \cdot 2^x - x^2 = 2^x$. The exponential 2^x is *invariant* under the operation of Δ.

Problem 2. Again, we simply apply the defintions.
(a) $x^{\underline{3}} = x(x-1)(x-2) = x^3 - 3x^2 + 2x$.
(b) $9^{\underline{4}} = 9 \cdot 8 \cdot 7 \cdot 6 = 3024$.
(c) $\Delta x^{\underline{2}} = \Delta(x^2 - x) = (x+1)^2 - (x+1) - (x^2 - x) = x^2 + 2x + 1 - x - 1 - x^2 + x = 2x$.

Problem 3. Recall that recursive defintions must have a base case and a recursive formula that computes each value in terms of previous values. So we define:

$$x^{\underline{n}} = \begin{cases} 1 & n = 0 \\ x^{\underline{n-1}} \cdot (x - n + 1) & n > 0 \end{cases}.$$

Problem 4. We use induction. If $n = 0$ then $x^{\underline{m+n}} = x^{\underline{m}} = x^{\underline{m}} \cdot 1 = x^{\underline{m}} x^{\underline{0}} = x^{\underline{m}} x^{\underline{n}}$.

Then, inductively assuming the formula is correct for $n-1$, we compute $x^{\underline{m+n}} = x^{\underline{m+(n-1)+1)}} = x^{\underline{m+(n-1)}}(x - (m + (n-1)))$ by the recursive definition. Then the inductive assumption tells us this equals $x^{\underline{m}}(x-m)^{\underline{n-1}}(x - (m + (n-1)))$. The last term in this product can be rewritten as $((x-m) - n + 1)$, and $(x-m)^{\underline{n-1}}((x-m) - n + 1)) = (x-m)^{\underline{n}}$ by the recursive definition. Substituting this into the expansion for $x^{\underline{m+n}}$ then gives the desired result.

Problem 5. We use induction again. When $n = 0$ then both sides of the proposed formula evaluate to 1 so the proposition is verified in the base case.

Now, inductively assume the formula is true for $n - 1$. Then $(-x)^{\underline{n}} = (-x)^{\underline{1+n-1}} = (-x)^{\underline{1}}(-x-1)^{\underline{n-1}}$ by the result of problem 4. From the base case, we know that the first factor is simply $-x$, while the inductive assumption tells us the second factor is $(-1)^{n-1}(x + 1 + n - 2)^{\underline{n-1}}$. Multiplying these together gives $(-1)^n(x+n-1)^{\underline{n-1}} \cdot x$. Of course, x can be rewritten as $(x + n - 1) - n + 1$. Substituting this into the previous expression and using the recursive defintion of $(x+n-1)^{\underline{n}}$ we obtain the desired result.

(There are certainly other ways to arrive at this result. A direct induction, not using the factorial law of exponents will work, though the notation gets even messier than the above. Using the relationship between falling factorials of x and rising factorials of $-x$ is also a good approach—*if* you remembered to properly define rising factorials and prove that relationship as the questions packet noted.)

Problem 6. Actually, as stated the proposition is not quite true when $n = 0$. That is because $x^{\underline{-1}}$ has not (yet) been defined, so technically $0 \cdot x^{\underline{-1}}$ is also undefined. (If you wrote this, you will receive full credit for this problem; you will also earn full credit for the following.) On the other hand, whatever $x^{\underline{-1}}$ is, clearly multiplying it by zero will produce zero. Of course, since $x^{\underline{0}} = 1$ is constant, problem 1(a) tells us that $\Delta x^{\underline{0}}$ is zero also, so the proposition is proven if $n = 0$.

Now if $n > 0$, we can directly compute $\Delta x^{\underline{n}} = (x+1)^{\underline{n}} - x^{\underline{n}} = (x+1)^{\underline{1+n-1}} - x^{\underline{n-1+1}}$. Now apply the factorial law of exponents to the first term and the recursive definition to the second term to obtain $(x+1)^{\underline{1}}(x+1-1)^{\underline{n-1}} - x^{\underline{n-1}}(x-n+1)$. Since $(x+1)^{\underline{1}} = x+1$ and $x+1-1 = x$ we can simplify this expression to $(x+1)x^{\underline{n-1}} - (x-n+1)x^{\underline{n-1}}$ and factoring out the $x^{\underline{n-1}}$ leaves the desired result.

Problem 7. We mirror the definition from problem 3:

$$x^{\underline{-n}} = \begin{cases} \dfrac{1}{x+1} & n = 1 \\ \dfrac{x^{\underline{-(n-1)}}}{x+n} & n > 1 \end{cases}.$$

Problem 8.

(a) If $n = 1$ then the statement is true by the definition of $x^{\underline{-1}}$. Now assume the statement is true for some positive $n-1$. Then by definition, $x^{\underline{-n}} = x^{\underline{-(n-1)}}/(x+n)$. Using the inductive assumption this becomes $\dfrac{1}{(x+n-1)^{\underline{n-1}}(x+n)}$. Writing $(x+n) = (x+n)^{\underline{1}}$ and applying the factorial law of exponents then shows this last simplifies to $1/(x+n)^{\underline{n}}$ as required.

Of course, the statement remains true if n is negative or zero, it just requires a few more steps to prove.

(b) First, the proposition is already proven if neither m nor n is negative. So we'll show the cases where both are negative or where just one is negative. So let $m = -a$ and $n = -b$ both be negative, so that a and b are positive. Then $x^{\underline{m+n}} = x^{\underline{-(a+b)}} = \dfrac{1}{(x+a+b)^{\underline{a+b}}}$ by part (a). As proven in problem 4, we can write this as $\dfrac{1}{(x+a+b)^{\underline{b}}(x+a)^{\underline{a}}}$. Using part (a) again gives us $(x+a)^{\underline{-b}}x^{\underline{-a}} = (x-m)^{\underline{n}}x^{\underline{m}}$.

Next, let m be positive, n be negative, and let $m = -n + k$ where $k \geq 0$ so that $m+n = k$. Then the left-hand side of the proposed identity simplifies to $x^{\underline{k}}$. The right-hand side is $x^{\underline{-n+k}}(x+n-k)^{\underline{n}}$. Since both $-n$ and k are positive, we can apply the result in problem 4 to split the first term into two parts, yielding $x^{\underline{k}}(x-k)^{\underline{-n}}(x+n-k)^{\underline{n}}$. Applying the previous part of this problem to the middle factor in this expression turns that factor into $\dfrac{1}{(x+n-k)^{\underline{n}}}$ and this cancels the third factor leaving only the first term in the product, $x^{\underline{k}}$ as desired.

The proof when m is positive (or zero), n neagitve but larger in absolute value than m is nearly identical, by writing $n = -m-k$ where $k \leq 0$. Now the left-hand side of the formula is $x^{\underline{-k}}$. The right-hand side is $x^{\underline{m}}(x-m)^{\underline{-m-k}}$. Then we apply the rule for negative falling exponents from part (a) to the second term, and the result of problem 4 to the resulting denominator. Applying the rule in part (a) and the result of problem 4 to the terms in the denomintor will now give the desired result, just as in part (a).

(c) We proceed in a manner similar to the proof in problem 6. $\Delta x^{\underline{-n}} = (x+1)^{\underline{-n}} - x^{\underline{-n}} = \dfrac{1}{(x+n+1)^{\underline{n}}} - \dfrac{1}{(x+n)^{\underline{n}}} = \dfrac{x+1}{(x+n+1)^{\underline{n}}(x+1)} - \dfrac{x+n+1}{(x+n+1)(x+n)^{\underline{n}}}$. Both denominators simplify to $(x+n+1)^{\underline{n+1}}$ so the whole expression becomes $\dfrac{-n}{(x+n+1)^{\underline{n+1}}} = -nx^{\underline{-n-1}}$.

Problem 9. Similar to the other recursive problems we've seen, we need

$$S_n f(x) = \begin{cases} f(0) & n = 1 \\ S_{n-1}f(x) + f(n-1) & n > 1 \end{cases}.$$

Problem 10. This is $0+1+2+\cdots+999$. Using the summation formula for an arithmetic series yields $999 \cdot 1000/2 = 499500$.

Problem 11. We proceed by induction. Since $\Delta f(x) = f(x+1) - f(x)$, when $n = 1$ we obtain $S_1 \Delta f(x) = f(0+1) - f(0) = f(n) - f(0)$.

Now assume that $S_{n-1}\Delta f(x) = f(n-1) - f(0)$. Then $S_n \Delta f(x) = S_{n-1}\Delta f(x) + \Delta f(n-1) = f(n-1) - f(0) + f(n) - f(n-1) = f(n) - f(0)$.

Problem 12. While we could do a full induction, if we are smart we can use the result of problems 6 and 7(c) to note that $x^{\underline{k}} = \frac{1}{k+1}\Delta x^{\underline{k+1}}$ and then our result follows from problem 11 and linearity (to factor the constant $\frac{1}{k+1}$ out of the S_n).

Problem 13. One approach to this problem would be to multiply out all the falling factorials and set up a system of equations to find a, b, and c. In this case, though, a simpler approach is to substitute various values for x to obtain the equations. For instance, inserting $x = 0$ makes $0^3 = 0^{\underline{3}} = 0^{\underline{2}} = 0^{\underline{1}} = 0$ so we learn immediately that $c = 0$. Trying $x = 1$ gives $1^3 = 1$, $1^{\underline{3}} = 1^{\underline{2}} = 0$ and $1^{\underline{1}} = 1$ so $b = 1$. Trying $x = 2$ makes $2^3 = 8, 2^{\underline{3}} = 0, 2^{\underline{2}} = 2^{\underline{1}} = 2$ so $a = 3$. Thus, $x^3 = x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}}$.

Problem 14. We are being asked to compute $S_n x^3$. Using the result of problem 13, this becomes $S_n(x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}})$. Now we can use problem 12 and linearity to evaluates this as $\frac{1}{4}n^{\underline{4}} + n^{\underline{3}} + \frac{1}{2}n^{\underline{2}}$. (Note, all the falling factorials evaluated at zero yield zero, so are omitted.)

Problem 15. This is just brute force arithmetic on the previous answer: $\frac{1}{4}n^{\underline{4}} + n^{\underline{3}} + \frac{1}{2}n^{\underline{2}} = \frac{1}{4}(x(x-1)(x-2)(x-3) + 4x(x-1)(x-2) + 2x(x-1) = \frac{1}{4}(x^4 - 6x^3 + 11x^2 - 6x + 4x^3 - 12x^2 + 8x + 2x^2 - 2x) = \frac{1}{4}(x^4 - 2x^3 + x^2)$, or, if you prefer it in factored form, $\frac{1}{4}x^2(x-1)^2$.

Problem 16. We are asked to find $S_n a^x$. From the formula for Δa^x we divide by $a-1$ to discover that $a^x = \frac{1}{a-1}\Delta a^x$. Applying S_n to this yields $S_n a^x = \frac{1}{a-1} S_n \Delta a^x = \frac{1}{a-1}(a^n - a^0) = \dfrac{a^n - 1}{a-1}$.

Problem 17. One last induction! The statement is clearly true when $n = 0$. So assume the statement true for $n-1$. Then for n we have

$$\begin{aligned}
(x+y)^{\underline{n}} &= (x+y)^{\underline{n-1}}(x+y-n+1) \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{\underline{k}} y^{\underline{n-1-k}}(x+y-n+1) \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{\underline{k}} y^{\underline{n-1-k}}((x-k) + (y-n+1+k)) \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} (x^{\underline{k}}(x-k) y^{\underline{n-1-k}} + x^{\underline{k}} y^{\underline{n-1-k}}(y-n+1+k) \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} (x^{\underline{k+1}} y^{\underline{n-1-k}} + x^{\underline{k}} y^{\underline{n-k}} \\
&= \sum_{k=0}^{n} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) x^{\underline{k}} y^{\underline{n-k}}
\end{aligned}$$

where, as per the usual convention, $\binom{n-1}{-1} = \binom{n-1}{n} = 0$. Of course, binomial coefficients satisfy the recurrence $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ which gives exactly the sum we are looking for.