Funny Factorials and Slick Sums

Spring 2017 ARML Power Contest

WARNING! In several of the problems, you may be tempted to use an ellipsis ("...") to shorten arguments or to indicate that a pattern continues. DO NOT do this. All proofs must be complete. Ellipsis probably means you need to use induction or recursion.

Let $f(x)$ be any function whose domain is the set $\mathbf{R}$ of real numbers. Create a new function $\Delta f(x)$ which is defined to be $\Delta f(x) = f(x+1) - f(x)$. So if, for instance, $f(x) = x^2$ then $\Delta f(x) = (x+1)^2 - x^2 = 2x + 1$. We will treat Δ as an operation that takes in functions and gives out functions—a kind of superfunction whose domain and range are regular functions.

Problem 1. For each function $f(x)$ below, compute and simplify $\Delta f(x)$.

[1] (a) $f(x) = c$, where c is constant.

[1] (b) $f(x) = ax + b$, a linear function.

[1] (c) $f(x) = x^3$.

[1] (d) $f(x) = 2^x$.

As part (c) of this problem demonstrated, Δ doesn't play particularly nicely with powers of x. But there is a variation of factorials that Δ does treat nicely. For any positive integer n, define $x^{\underline{n}}$ by the formula $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$. It's kind of a cross between the nth power and a factorial. The underline is meant to remind you that the things you are multiplying together keep decreasing. You can pronounce this as *x to the nth falling* or *the nth falling factorial of x*. It will be convenient to define $x^{\underline{0}} = 1$. There is a similar *rising factorial* $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$.

Problem 2. Expand and simplify each of the following.

[1] (a) $x^{\underline{3}}$.

[1] (b) $9^{\underline{4}}$.

[2] (c) $\Delta x^{\underline{2}}$.

[2] **Problem 3.** In keeping with the warning at the beginning of the problem, write a proper recursive defintion for $x^{\underline{n}}$ for all nonnegative integers n.

We can make a few observations about falling and rising factorials. For instance, if n is a nonnegative integer, then $n^{\underline{n}} = 1^{\overline{n}} = n!$, while if $m > n$ are both nonnegative integers then $n^{\underline{m}} = 0$. Also, $x^{\underline{n}} = (-1)^n(-x)^{\overline{n}}$ where the exponent on the (-1) is a regular power. While all these appear more-or-less obvious, if you want to use one of these facts in your solutions, you will—for completeness—need to prove it as a lemma.

We will treat falling and rising factorials like regular exponents with regard to order of operations. So, for instance, $(-5)^{\underline{3}} = (-5)(-6)(-7)$ while $-5^{\underline{3}} = -(5^{\underline{3}}) = -(5 \cdot 4 \cdot 3)$.

[2] **Problem 4.** Prove the factorial law of exponents: $x^{\underline{m+n}} = x^{\underline{m}} \cdot (x-m)^{\underline{n}}$ for all nonnegative integers m and n.

[2] **Problem 5.** Prove the symmetry law $(-x)^{\underline{n}} = (-1)^n(x+n-1)^{\underline{n}}$ for all nonnegative integers n. Note that the exponent on the (-1) is a regular exponent, not a factorial exponent!

You might have noticed that $\Delta x^{\underline{2}}$ from problem 2(c) was less messy than Δx^2 or Δx^3 that we calculated earlier. Falling factorials were created to make nice formulas with Δ.

[2] **Problem 6.** Prove that for any nonnegative integer n, $\Delta x^{\underline{n}} = nx^{\underline{n-1}}$.

We can define falling (and rising) factorials with *negative* exponents in a manner similar to the definition of negative powers. Note that to get from $x^{\underline{3}}$ to $x^{\underline{2}}$ we divide by $(x-2)$. Then we would have to divide by $(x-1)$ to drop the exponent to $x^{\underline{1}}$. And we divide again by $x = (x-0)$ to lower the exponent to $x^{\underline{0}}$. It seems that we should now divide by $(x+1)$ to get to $x^{\underline{-1}}$. That is, it looks like we should define

$$\begin{aligned} x^{\underline{-1}} &= \frac{1}{x+1}, \\ x^{\underline{-2}} &= \frac{1}{(x+1)(x+2)}, \\ x^{\underline{-3}} &= \frac{1}{(x+1)(x+2)(x+3)}, \end{aligned}$$

and so forth.

[1] **Problem 7.** If n is a positive integer, provide a recursive definition for $x^{\underline{-n}}$.

Problem 8.

[2] (a) If n is a positive integer, prove $x^{\underline{-n}} = \dfrac{1}{(x+n)^{\underline{n}}}$.

[2] (b) Prove that the law of exponents as in problem 4 continues to hold for arbitrary integers m and n.

[2] (c) Prove that the Δ-rule from problem 6 continues to hold for negative integers.

Let n be a positive integer and $f(x)$ a function. Define an operation S_n by $S_n f(x) = f(0) + f(1) + \cdots + f(n-1)$. Be careful to note that there are exactly n terms in this sum, running from 0 to $n-1$. For example, if $f(x) = 2^x$ then $S_5 f(x) = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 31$.

[2] **Problem 9.** Define $S_n f(x)$ recursively.

[1] **Problem 10.** Compute $S_{1000} x$.

[2] **Problem 11.** Prove, for any n and any function $f(x)$, that $S_n(\Delta f(x)) = f(n) - f(0)$.

You may assume—it is not difficult to prove—that both Δ and S_n, for any n, are *linear*. That is, given functions $f(x)$ and $g(x)$ and constants a and b, that $\Delta(af(x) + bg(x)) = a\Delta f(x) + b\Delta g(x)$ and $S_n(af(x) + bg(x)) = aS_n f(x) + bS_n g(x)$.

[2] **Problem 12.** Prove that $S_n x^{\underline{k}} = \dfrac{n^{\underline{k+1}} - 0^{\underline{k+1}}}{k+1}$ where k is any integer other than -1.

[2] **Problem 13.** Compute the values of constants a, b, and c so that $x^3 = x^{\underline{3}} + ax^{\underline{2}} + bx^{\underline{1}} + cx^{\underline{0}}$.

[2] **Problem 14.** Use the result of the previous problem to compute $0^3 + 1^3 + 2^3 + \cdots + (n-1)^3$ in terms of falling factorials.

[1] **Problem 15.** Expand the terms of your solution to the previous problem to express $0^3 + 1^3 + \cdots (n-1)^3$ as a polynomial in n.

It is similarly possible to easily obtain expressions for the sums of squares, fourth powers, or any other (positive integer) powers of x, as long as there is a convenient way to convert back and forth between polynomials and falling factorials. To do that efficiently, you need *Stirling numbers*, which might be a topic for a future ARML Power Contest.

[2] **Problem 16.** Note that $\Delta a^x = a^{(x+1)} - a^x = (a-1)a^x$. Use this to obtain a formula for the sum of a finite geometric series $1 + a + a^2 + a^3 + \cdots + a^{n-1}$.

[3] **Problem 17.** Prove that the falling factorials satisfy the binomial theorem: for nonnegative integers, n, $(x+y)^{\underline{n}} = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}$.