Cryptarithms—Solutions

Fall 2016 ARML Power Contest

Problem 1.
(a) 999, 99, 1098
(b) 1, 0
(c) Since the value of `ME` is less than 100, the value of `YOU` must be more than 900 for their sum to be a four-digit number. Since we know from part (b) that `O` must represent 0, and a digit can't be represented by more than one letter, the value of `YOU` is at most 908. Since the digit 9 is already used, we know that the value of `ME` is less than 90. So the sum `YOU` + `ME` is less than 998 so cannot be a four-digit number. So the puzzle has no solution.

Problem 2. So far we know that `O`, `M`, and `S` represent 0, 1, and 9 respectively, and that the pair `EN` represents 23, 34, 45, 56, 67, or 78. Note that the addition in the tens- and hundreds-places reverses these two digits. So in all of the possible cases, `NE` − `EN` = 9. Since the digit 9 is already represented by `S`, we see that `R` must be 8 and there must be a carry from the units-place. That also rules out the possibility that `N` represents 8 (so `E` cannot be 7).

We go through the remaining possibilities. If `E` is 2, then `D` would have to be 8 or 9 to cause a carry, but both those digits are already taken. If `E` is 3, a `D` of 7 would cause a carry, but then `Y` would be 0, which is already taken. Likewise, with an `E` of 4, `D` could be 6, 7, 8, or 9 to cause a carry, but 8 and 9 are already used, while 6 and 7 cause `Y` to be 0 or 1, which are already taken.

That leaves `E` as either 5 or 6. Check that 6 doesn't work (`D` can't be 0, 1, 2, or 3 because they cause no carry, 4 or 5 because they cause `Y` to be an already used value, 6 because that is the value of `E`, 7 because that is the value of `N` if `E` is 6, and of course 8 and 9 are ruled out). So the only hope is to try `E` representing 5; then `N` would be 6. `D` can then represent 7 and `Y` would be 2. The sum would be $9567 + 1085 = 10652$ which is true. So the solution is

0	1	2	3	4	5	6	7	8	9
O	M	Y			E	N	D	R	S

.

Problem 3. Start with the obvious: since the only possible carry into the largest position can be a one, `P` represents 1. And since there cannot be a carry into the units digit, we learn that `N` represents 0. Now look in the hundreds-place, where adding 0 to `U` resulted in `E`, meaning there had to be a carry and `E` is one larger than `U` (or `U` is 9 and `E` is 0—this doesn't happen since we already know that `N` is 0).

In the hundred-thousands-place, we have $\texttt{A} + \texttt{R} = \texttt{A}$ and the digit 0 is already taken, so R must represent 9 and there must be a carry coming into this position. Then, from the tens-place we learn that T is one less than U. Since U is not 9, there will be no carry from the hundreds-place into the thousands-place. Thus, the digits T and A must add to 10.

Since we know that U is one more than T and E is one more than U and the digits 0, 1, and 9 have already been taken, the trio TUE represents one of 234, 345, 456, 567, or 678. But T can't be 4, or A and E would both have to be 6. Similarly, if T were 5 then A would have to be 5 also.

Let's just try the remaining possibilities. Setting T to be 2 and U to be 3 runs causes a problem in the hundred-thousands-place where, even with the carry from the ten-thousands-place, the largest we could make S is 8 (9 is already taken!) which would force L to be 0, 1, or 2 (or there to be no carry into the millions place), all of which are already taken. A similar problem happens by setting T to be 3. The only remaining possibility is to set T to be 6. This leaves three choices for S, but two of them lead to L being 0 or 1. The remaining choice, 5, leads to the correct addition $546790 + 794075 = 1340865$ and the solution

0	1	2	3	4	5	6	7	8	9
N	P		L	A	S	T	U	E	R

.

Problem 4. The units-place of this sum tells us that $\texttt{Y} + \texttt{N} + \texttt{N} = \texttt{Y}$, perhaps with a carry. So N is either 5 or 0. The tens-place of the sum tells us that there can't be a carry (because $\texttt{E} + \texttt{E} + 1$ cannot add to zero or ten), so N must be 0. Then E will be 5.

There must be carries from both the hundreds-place and the thousands-place. The three digits in the hundreds-place can only add to numbers in the teens or twenties, so the carry into the thousands-place must be a 1 or a 2. If it is a 1, that would make O a 9 since we need a carry from this column also. But if O is 9 and the carry is only one, that would make I be 0, which has already been used. That means that the carry must be 2 and O must be 9, and I is 1.

In the tens-place, $\texttt{T} + 5 + 5 = 10 + \texttt{T}$, so the carry into the hundreds-place is a 1. So, since the carry into the thousands-place is 2, we know that $1 + \texttt{R} + 2\texttt{T}$ is at least 20. In fact, since 0 and 1 are already taken, X must be at least 2, so $1 + \texttt{R} + 2\texttt{T}$ is at least 22. Since R is not the 9, this last sum requires T to be either 7 or 8.

Trying 7 for T means R must be 8, and X will be 3. The only digits now left are 2, 4, and 6, but F and S must be consecutive, so this scenario is impossible. Therefore T must be 8.

We still must have $1 + \texttt{R} + 2\texttt{T}$ be at least 22, so R will now be either 6 or 7. But choosing it to be 6 again requires X to be 3, which has the same problem as above. The only choice left is R is 7, X is 4, and we can finish the puzzle with F being 2, S being 3, and Y being 6. The complete solution is

0	1	2	3	4	5	6	7	8	9
N	I	F	S	X	E	Y	R	T	O

.

Problem 5. The number THREE is a square. The squares of numbers from 200 and up are either six-digit numbers, or do not begin with the same digit as the number of which they are the square. So the T represents the digit 1. Further, the only squares that end in a repeated digit are those that end in -00 or -44. Since O and E can't both be 0, the former is ruled out. The numbers less than 200 whose squares end in -44 are $112^2 = 12544$ (but T and W can't both represent the digit 1), $138^2 = 19044$ (which is the answer), $162^2 = 26244$ (which doesn't begin with a 1), and $188^2 = 35344$ (ditto), so we have the answer

0	1	2	3	4	5	6	7	8	9
R	T		W	E				O	H

.

Problem 6. First, the digit E must be 1 or 2. Leading digits can't be 0, and larger values make the sum larger than 10000.

Now look at the hundreds-place. We need four copies of I plus any carry from the tens-place to add to a number that ends in I. That is, we need $4\mathtt{I} + \text{carry} \equiv \mathtt{I} \pmod{10}$. The carry can be 0, 1, 2, or 3. So we learn that the value of I can be 0 (if the carry is 0), 3 (if the carry is 1), 6 (if the carry is 2), or 9 (if the carry is 3).

Let's look again at the value of E. If it is 2, then I cannot be 6 or 9, or the sum will be more than 10000. If I is 3, we know that the carry from the tens-place must have been a 1. Looking in the tens-place, then, we see that if c is the carry from the units-place, we have $4\mathtt{N} + c = 12$. Since the carry c can only be 0, 1, 2, or 3, the only solution is that N is 3. But we can't have both N and I representing 3, so if E is 2 then I being 3 is ruled out.

So if E is 2, the only possibility left is that I is 0. This requires zero carry from the tens-place, so N is 0, 1, or 2. Since 0 and 2 are already taken, N would have to be 1. But then the addition in the tens-place will not work out (the carry from the units-place can't be larger than 3). We conclude that E represents 1.

Looking at the tens-place we find that N must be 0 or 5 (with a carry of 1 from the units-place), or 2 or 7 (with a carry of 3 from the units-place). Now we examine the possibilities one by one. First, if I represents 0, then there can be no carry from the tens-place, which would also require N to be 0. Nor can I represent 6, because now EINS is between 1602 and 1698, so the sum of four copies of EINS would add to a number between 6408 and 6792, requiring V to also represent 6.

Trying 9 for I means that the carry from the tens-place had to have been 3. That means that N would have to be 7. But now EINS is between 1970 and 1978, so quadrupling it leads to a number between 7880 and 7912, thus requiring V to also represent 7.

The only remaining possibility is to have I represent 3. As we have seen, this requires a carry of 1 from the tens-place, which means that N will have to represent 2. The tens-place now requires a carry of 3 from the units-place, meaning S represents either 8 or 9. We rule out 8 because that would result in R representing 2 which is already represented by N. So S represents 9, and we have the sum $1329 + 1329 + 1329 + 1329 = 5316$. The answer grid is thus

0	1	2	3	4	5	6	7	8	9
	E	N	I		V	R			S

.

Problem 7. Let m be the three-digit number represented by BIG and n the number represented by TOP. Then the equation tells us $9(1000m+n) = 4(1000n+m)$, or $8996m = 3991n$. We divide each side by the common factor of 13 are we are left with $692m = 307n$. The coefficients are relatively prime, so we must have $m = 307$ and $n = 692$, giving us the answer

0	1	2	3	4	5	6	7	8	9
I		P	B			T	G		O

.

Problem 8. First, notice that K must be 1, 2, or 3, or the square of KISS will have eight digits, not seven.

Let's concentrate on the digit represented by S. It can't be 0, because then $(\mathtt{KISS})^2$ would also end in 0, but N can't represent 0 if S does. The same is true for 1, 5, and 6.

Now notice that the hundreds-digit of both numbers is I. So let's see what happens when we square a number ending in ISS.

If S is 2, then $(\mathtt{ISS})^2$ ends in $(4\mathtt{I} + 4)84$. In other words, whatever digit I represents, quadruple it and add four to get the hundreds-place of the square. Or rather the units-digit of this number is the hundreds-place of the square. The only choice for I for which this results in the same digit is 2, but we are already assuming S is 2, so this is ruled out, and S cannot be 2. In a similar way, no choice of I works if S is 4 or 7. If S is 8, note that the square of any number ending in 88 will itself end in 44, but the last two digits of PASSION are not the same, so 8 is also ruled out. That implies S is either 3 or 9.

When S is 3, several choices of I will work out. If I is 0, K can be 1 or 2, and we try $1033^2 = 1067089$ and $2033^2 = 4133089$. Notice that the first of these does not have the form PASSION because the hundreds- and hundred-thousands- places are the same. But the second does work. Just to be complete, we check that the other possibilities do not work.

If S is 3, I could be 2 and K could be 1, leading to $1233^2 = 1520289$. If I is 4, K could be 1 or 2, but $1433^2 = 2053489$ and $2433^2 = 5919489$. We could also try I as 6, but $1633^2 = 2666689$ and $2633^2 = 6932689$. The other possibilities occur when S is 9, and then I must be 6, leading to trying $1699^2 = 2886601$ and $2699^2 = 7284601$. Of all these possibilities, only one had the form of the word `PASSION`, so that must be our number, and our final answer is

0	1	2	3	4	5	6	7	8	9
`I`	`A`	`K`	`S`	`P`				`O`	`N`

.

Problem 9. So that we can refer to the individual digits in this problem more easily, let's rewrite the problem as $\texttt{abc} \times \texttt{de} = \texttt{fghi} + \texttt{jklm0} = \texttt{nopqr}$ where each letter stands for a digit, digits can be repeated, and 0 has been inserted into the second addend to account for the left shift in the multiplication process.

Keep in mind that each digit must be prime, so each is 2, 3, 5, or 7. Also note that the product of pairs of these are among the numbers 4, 6, 10, 14, 9, 15, 21, 25, 35, and 49. The only ones of these that end in a prime digit are 15, 25, and 35. This tells us that at least one of our multipliers must end in a 5 and the other is odd (it could also be 5, but only one needs to be 5).

Expanding the logic of the previous paragraph, if the three-digit multiplier doesn't end with a 5, then the two-digit multiplier must be 55. Once you know that digit `e` is 5, then `a` can't be 2 or 3 (else `f` is 1) leaving very few possibilities for `abc`. In fact, the only two possibilities for `abc` that leave all prime digits in `fghi` are 555 and 755. But multiplying $555 \times 55 = 30525$ and $755 \times 55 = 41525$, and neither of these products consists of all prime digits.

Now look at the digit `e`. It certainly can't be 2 (or else `i` would be 0). It can't be 7, because then the carry in computing `h` would be 3, which would require `b` to be 7, and then the carry into the computation of `g` would be 5, and no number would work for `a` to keep all the digits of `fghi` prime.

We've already seen that if `e` is 5 the only possibilities for `abc` are either 555 or 755. We quickly check the products of these two with 25, 35, 55, and 75 and conclude that `e` cannot be 5. Thus, `e` must be 3.

Once we know that `c` is 5 and `e` is 3, we learn that `b` must be either 2 or 7 if we want digit `h` to be prime. Note that `b` can't be 2, because then there would be no carry into the hundreds digit, making it impossible to make both `f` and `g` prime (alternately, multiplying each of 225, 325, 525, and 725 by 3 either produces only a three-digit number or a number that does not have all prime digits). Hence, the digit `b` must be 7. By multiplying 275, 375, 575, and 775 by 3, we then learn that `a` must be 7. So we have found that the number `abc` is 775.

Now we just need to investigate the digit `d`. Quickly checking 775 times 2, 3, 5, and 7 leads us to the conclusion that the only possible digit for `d` is 3.

So our multiplication is $775 \times 33 = 25575$. Written out, the complete answer is:

```
    7 7 5
*     3 3
---------
  2 3 2 5
2 3 2 5
---------
2 5 5 7 5.
```

Problem 10. Let's again rewrite this as `abc` $\times$ `de` $=$ `fghi` $+$ `jkl0` $=$ `mnopq`. Since `abc` is greater than 200, and `abc` $\times$ `d` $=$ `jkl` which is less than 1000, we find that `d` must be less than 5. Note that `d` cannot be 1, since then `abc` and `jkl` would be equal, but `b` is even while `k` is odd. So the only odd number left for `d` is 3.

This in turn tells us that `a` must be 2, since all larger even numbers lead to a four-digit product with `d`. So `abc` is less than 300, and its product with `e` will be less than 3000, so `f` must be 2. Since `f` $+$ `j` must produce a carry, we learn that `j` $= 8$. This forces `b` to be 8 (neither $267 \times 3 = 801$ nor $269 \times 3 = 807$ has the form `EOO`) and then we quickly learn that the only choice for `c` that works is 5. So `abc` is 285.

Lastly, since $285 \times 7 = 1995$ is too small to be `fghi`, we learn that `e` is 9. Putting all the pieces together yields the solution:

```
    2 8 5
*     3 9
---------
  2 5 6 5
  8 5 5
---------
1 1 1 1 5.
```

Problem 11. Once again, we rename the unknown entries, so that the problem becomes `abc` $\times$ `de` $=$ `fghi` $+$ `jkl0` $=$ `mnop`. Since `d` is at least 2, we find that `a` is either 1 or 3. But if `a` is 1, then `abc` is less than 200, and its product with `e` is less than 2000, but since `f` is even, we have `fghi` is at least 2000, a contradiction. So `a` is 3.

Now since `abc` is between 300 and 400, and its product with the even digit `d` is less than 1000, we know that `d` must be 2. Also, since `abc` is less than 400 but its product with `e` is larger than 2000, we know that `e` is larger than 5. Since `e` must be even, it will be 6 or 8.

The digit `b` must be 0, 2, or 4 (otherwise, doubling `abc` would lead to a number between 700 and 800, but we know its first digit must be even). But it can't be 0, because then `abc` is between 300 and 310. Multiplying those numbers by 6 yield numbers between 1800 and 1860, while multiplying by 8 gives 2400 to 2480, so neither choice of `e` would give an acceptable result for `fghi`.

Likewise, `c` must be 6 or 8 to allow the digit `k` to be odd. So we have that `abc` is one of 326, 328, 346, or 348, while `de` is either 26 or 28. It's not hard to try the eight combinations, and learn that only one makes all the rest of the digits have the correct parities. The answer is:

```
  3 4 8
*   2 8
-------
2 7 8 4
6 9 6
-------
9 7 2 4.
```

Problem 12. The only base in which the alphametic has a solution is base 9. We can prove this as follows.

Let b be a base in which the problem has a solution. Since there are 4 different letters which appear in the alphametic, there are at least 4 different digits in base b, so $b \geq 4$. Number the columns in the addition problem from left-to-right, so that column 1 reads `K + K + K = T` and so forth.

Consider the fifth column, where we have 3`O` $=$ `O`, perhaps with a carry. We conclude that `O` is either 0 or $b/2$. The carry from the fourth column is either 0, 1, or 2, so if `O` is $b/2$ then `K` is $b/2 + c$ where c is this carry. This will always be larger than $b/2$ unless $b = 4$ and the carry is 2, in which case `K` would have to be 0, which is impossible because leading zeroes are forbidden. But if `K` is larger than $b/2$ then the first column addition would produce a carry, which it does not. So we conclude that `O` is 0.

Now we know that `K` is just the carry from the fourth column, and is not 0, so `K` is either 1 or 2. There can be no carry from the third column into the second, so the second column tells us that 3`Y` is either 0, b, or $2b$. It can't be 0 because `Y` is not 0. So it is either b or $2b$. Either way, we learn that b is divisible by 3. Also, the carry from the second column into the first is either one or two. Since `K` is known to be 1 or 2, we find there are four possible values for `T`: 4, 5, 7, or 8.

Now we plow through the possibilities. If `K` is 1 and `T` is 4, the third and fourth columns tell us that $b +$ `Y` $= 12$. Since $0 <$ `Y` $< b$ and b is divisible by 3, the only possibility here is $b = 9$ and the sum is $13040 + 13040 + 13040 = 40130$ which is true in base 9.

If `K` is 1 and `T` is 5, we find that $b +$ `Y` $= 15$. So either $b = 9$ and `Y` $= 6$ with solution $16050 + 16050 + 16050 = 50160$ which is true in base 9, or we try $b = 12$ and `Y` $= 3$. But then the addition in the second column fails, so this is not a possibility.

Next, try $\texttt{K} = 2$ and $\texttt{T}$ being 7. Then the third and fourth columns give us $2b + \texttt{Y} = 21$. Since $\texttt{Y}$ is still a base-b digit and is not 0 and b is divisible by 3, the only solution here is $b = 9$ and $\texttt{Y} = 3$. This leads to the (correct in base 9) equation $23070 + 23070 + 23070 = 70230$.

Finally, with $\texttt{K}$ set to 2 and $\texttt{T}$ set to 8, the third and fourth columns give $2b + \texttt{Y} = 24$. The only hope for a solution this time is $b = 9$ and $\texttt{Y} = 6$, which does work to give us $26080 + 26080 + 26080 = 80260$ which is also a solution.

So in every case the only base in which we could find a solution is 9, and there were four solutions.