2017 ARML Local Problems
Team Round (45 minutes)

T-1 A fair six-sided die has faces with values 0, 0, 1, 3, 6, and 10. Compute the smallest positive integer that *cannot* be the sum of four rolls of this die.

T-1 Solution The numbers between 1 and 23, inclusive, can be the sum of at most 3 non-zero die rolls $(1, 1+1, 3, 3+1, 3+1+1, 6, 6+1, 6+1+1, 6+3, 10, 10+1+1, 6+6, 10+3, 10+3+1, 6+6+3, 10+6, 10+6+1, 6+6+6, 10+6+3, 10+10, 10+10+1, 10+6+6, 10+10+3)$. The numbers 1 through 33 inclusive can be achieved via four rolls by adding an additional roll of 10 to the up to three rolls that sum to 14 to 23. Accordingly, $\boxed{34}$ is the smallest positive integer that cannot be the sum of four rolls of this die.

T-2 For a positive integer k, let z_k be the number of terminal zeroes of the product $1!2!\cdots k!$. For example, $z_6 = 2$ because $1!2!3!4!5!6! = 24883200$. Compute z_{100}.

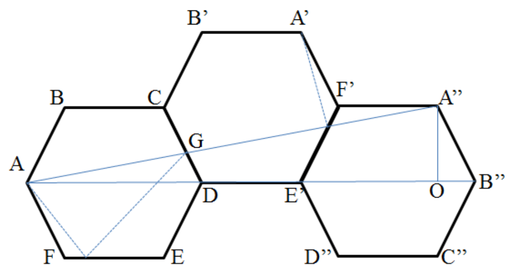
T-2 Solution If each factorial is expanded out to its individual terms, there would be 96 5s, 91 10s, ..., 6 95s, and 1 100 terms. Each multiple of five term adds one zero to the end of z_{100}, with each multiple of 25 contributing an additional zero to the end of z_{100}. The total number of zeros is $(96 + 91 + \cdots + 6 + 1) + (76 + 51 + 26 + 1) = \boxed{1124}$.

T-3 Let $ARML$ be a square of side length 5. A point B on side $\overline{MR}$ and C on side $\overline{ML}$ are selected uniformly at random and independent of one another. Compute the expected area of triangle ABC.

T-3 Solution Put $A = (0, 0)$, let $B = (5, y)$ and $C = (x, 5)$. Then the area of ABC is equal to $\frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 5 & y \\ 1 & x & 5 \end{vmatrix} = \frac{1}{2}(25 - xy)$. Since x and y are independent, $E[xy] = E[x]E[y] = (5/2)^2 = 25/4$. The expected area of ABC is $\frac{1}{2}\left(25 - \frac{25}{4}\right) = \boxed{75/8}$.

T-4 The edges of regular hexagon $ABCDEF$ are made of mirrors. A laser is fired from A toward the interior of edge $\overline{CD}$, striking it at point G. The laser beam reflects off the interior of exactly one additional edge and returns to A. Compute $\tan(\angle DAG)$.

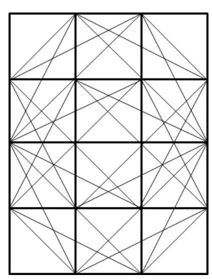
T-4 Solution Reflect the hexagon $ABCDEF$ across the segment $\overline{CD}$ to get the hexagon $A'B'CDE'F'$. In order to strike exactly one additional side and return to A, the laser should strike $\overline{EF}$ next before returning to A. Reflecting the second hexagon about $\overline{E'F'}$ to get the hexagon $A''B''C''D''E'F'$, the segment $\overline{AA''}$ traces the path of the laser beam when reflected back upon the initial hexagon as shown below.



Dropping the perpendicular from A'' to $\overline{AB''}$ at O, $AO = \frac{9}{2}AB$ and $A''O = \frac{\sqrt{3}}{2}AB$, and $\tan(\angle DAG) = \tan(\angle OAA'') = \frac{\sqrt{3}/2}{9/2} = \boxed{\frac{\sqrt{3}}{9}}$.

T-5 Let S be the set of lattice points $\{(x, y) : x, y \in \mathbb{Z}, 0 \le x \le 3, 0 \le y \le 4\}$. Compute the number of subsets of S of 4 lattice points that form the vertices of a square.

T-5 Solution In the 3×4 grid, there are 12 squares with side length 1, 6 of side length 2, and 2 of side length 3. There are also squares with edges not aligned with the grid; 6 of side length $\sqrt{2}$ and 4 of side length $\sqrt{3}$, for a total of $\boxed{30}$.



T-6 If A is an acute angle such that $\sin 15° + \cos 15° = \sqrt{2} \sin A$, compute $\cos A$.

T-6 Solution Multiplying both sides of the equation by $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2} \sin 15° + \frac{\sqrt{2}}{2} \cos 15° = \sin A \to \sin A = \cos 45° \sin 15° + \sin 45° \cos 15° = \sin(45° + 15°) = \sin(60°)$. Since A is acute, $A = 60°$, so $\cos A = \boxed{\frac{1}{2}}$.

T-7 Consider the following algorithm for a given non-negative integer n:
Step 1: Initialize $k = 1$.
Step 2: Replace n with the product of its digits.

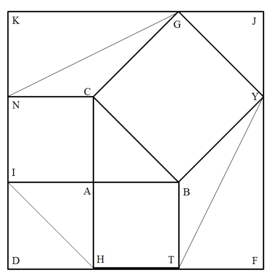
Step 3: If $n \leq 9$ return k and exit. If $n > 9$, increment k by 1 and return to Step 2.

For example, if $n = 125$, then the algorithm returns 2 because 125 is replaced by $1 \times 2 \times 5 = 10$ which is replaced by $1 \times 0 = 0$, at which point the algorithm returns 2 and exits. Compute the smallest value of n such that this algorithm returns 3.

T-7 Solution For this process, all integers up to 24 return 1. The integers 25 to 29 return 2. The smallest integer for which the product of its digits is between 25 and 29, inclusive, is $\boxed{39}$.

T-8 Three co-planar squares, $BAHT$, $CAIN$, and $BCGY$ have areas 16, 16, and 32, respectively. If the squares only intersect pairwise at the vertices A, B, and C, compute the area of the convex hexagon $THINGY$.

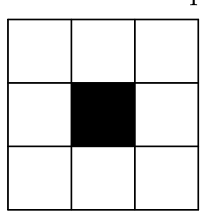
T-8 Solution As the three squares have side lengths of 4, 4, and $4\sqrt{2}$, it is clear that ABC is an isosceles right triangle. Let $DFJK$ be the square of side length 12 for which $\overline{HT}$ is a subsegment of $\overline{DF}$ and $\overline{IN}$ is a subsegment of $\overline{DK}$. Then $[THINGY] = [DFJK] - [HID] - [FTY] - [JGY] - [GNK] = 144 - 8 - 16 - 8 - 16 = \boxed{96}$.



T-9 Compute the coefficient of x^8 in the expansion of $(x^2+x+1)^8$ after combining like terms.

T-9 Solution There are five ways to represent x^8 as the product of exactly eight terms of the form x^2, x, and 1: $(x^2)^4(x)^0(1)^4$, $(x^2)^3(x)^2(1)^3$, $(x^2)^2(x)^4(1)^2$, $(x^2)^1(x)^6(1)^1$, and $(x^2)^0(x)^8(1)^0$. The coefficient of x^8 will be $\binom{8}{4,0,4} + \binom{8}{3,2,3} + \binom{8}{2,4,2} + \binom{8}{1,6,1} + \binom{8}{0,8,0} = \frac{8!}{4!0!4!} + \frac{8!}{3!2!3!} + \frac{8!}{2!4!2!} + \frac{8!}{1!6!1!} + \frac{8!}{0!8!0!} = 70 + 560 + 420 + 56 + 1 = \boxed{1107}$.

T-10 Consider the following 3×3 grid with its center square removed, as shown.



Compute the number of distinct ways to fill in the grid with the integers 1 through 8, each appearing exactly once, such that:

– In each of the three rows, the entries are increasing from left to right.

– In each of the three columns, the entries are increasing from top to bottom.

T-10 Solution First place 1 and 8 in the upper-left and bottom-right. WLOG 2 is the entry to the right of 1 (by symmetry the final answer is twice the number of configurations with 2 to the right of 1). Then, the grid is filled in as follows:

1	2	a
x	■	b
y	z	8

Observe that $x \le y \le z$, $a \le b$, and $x \le b$. There are $\binom{5}{2}$ ways to pick a and b, and all of them result in valid configurations except $(a, b) = (3, 4)$. Hence, the final answer is $2\left(\binom{5}{2} - 1\right) = \boxed{18}$.

T-11 Kevin and Tarsha are playing a game with four fair standard six-sided dice. The four dice are rolled and if the numbers appearing on top of the four dice are all different, Kevin pays Tarsha \$1. If not, Tarsha pays Kevin \$$k$ dollars. Compute the value of k such that the game is fair, in other words, the expected value of each play of the game to both players is zero.

T-11 Solution Rolling the dice in sequence, it is clear to see that the probability that all four die rolls are different is $\frac{6\times5\times4\times3}{6\times6\times6\times6} = \frac{5}{18}$. The game is fair provided $\frac{5}{18}(1) + \frac{13}{18}(-k) = 0 \to k = \boxed{\dfrac{5}{13}}$.

T-12 Compute the minimum value of ab such that $\log_2(a^4b^{-3}) = 3$ and $\log_2(a^4b^3) = 9$.

T-12 Solution Note that while b must be positive, a may be negative. The two equations in the problem are equivalent to $2\log_2(a^2) - 3\log_2(b) = 3$ and $2\log_2(a^2) + 3\log_2(b) = 9$. Adding the two equations together gives $4\log_2(a^2) = 12 \to \log_2(a^2) = 3 \to \log_2(b) = 1$. Thus, $b = 2$ and $a = \pm 2\sqrt{2}$, so the minimum value of ab is $\boxed{-4\sqrt{2}}$.

T-13 Compute

$$\sum_{n=3}^{10} \frac{20}{(n-2)(n+2)}$$

.

T-13 Solution $\frac{20}{(n-2)(n+2)} = \frac{5}{n-2} - \frac{5}{n+2}$, so

$$\begin{aligned}
\sum_{n=3}^{10}\frac{20}{(n-2)(n+2)} &= 5\left(\sum_{n=3}^{10}\frac{1}{n-2}-\sum_{n=3}^{10}\frac{1}{n+2}\right)\\
&= 5\left(\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{8}\right)-\left(\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{12}\right)\right)\\
&= 5\left(\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)-\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}\right)\right)\\
&= 5\left(\frac{1980+990+660+495}{1980}-\frac{220+198+180+165}{1980}\right)\\
&= 5\left(\frac{4125}{1980}-\frac{763}{1980}\right)=\frac{3362}{396}=\boxed{\frac{1681}{198}}.
\end{aligned}$$

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T-14 The six-digit number 142857 has the property that moving the rightmost digit to the left of the number results in multiplying the number by five ($142857 \times 5 = 714285$). Compute the smallest six-digit number with the property that moving the rightmost digit to the left of the number results in multiplying the number by four.

T-14 Solution Let N be the six-digit number with the property described in the problem. Let a_0 be the rightmost digit of N such that $N = 10k + a_0$, where $10000 \leq k \leq 99999$. Therefore, $4N = 10^5 a_0 + k$, and combining the two equations gives $4(10k + a_0) = 10^5 a_0 + k \rightarrow 39k = 99996a_0 \rightarrow k = 2564a_0$. k is greater than or equal to 10000 for the first time when $a_0 = 4$, and the resulting six-digit number is $\boxed{102564}$.

T-15 Compute the number of distinct sequences $(x_1, x_2, \ldots, x_{14})$ with the following properties:

- $x_k \in \{1, 2, \ldots, 14\}$ for each $k = 1, 2, \ldots, 14$.
- The number $20x_{k+1}^2 - 17x_k^2$ is divisible by 14 for each $k = 1, 2, \ldots, 13$.

T-15 Solution By the Chinese Remainder theorem it suffices to look modulo 2 and modulo 7. Modulo 2, there are clearly only two solutions since x_k must be even for $k = 1, 2, \ldots 13$.

Modulo 7, there are two possible situations:

- The values of $x_k^2 \pmod 7$ cycle between 1, 4, 2 in that order. There are 3 ways to choose the value x_1 modulo 7, and afterwards for each k there are 2 ways to choose x_k based on x_{k-1}. Hence $3 \cdot 2^{14}$ sequences in this case.
- The value $x_k^2 \pmod 7$ is always zero. There is only 1 sequence in this case, the constant zero sequence.

So the final answer is $2\,(3 \cdot 2^{14} + 1) = \boxed{98306}$.

Individual Round (10 minutes per pair)

I-1 A positive integer m is *stable* if $m = 2^n - n^2$ for some positive integer n. Compute the number of stable positive integers less than 2017.

I-1 Solution Let $f(n) = 2^n - n^2$. The initial values of $f(n)$ are $f(1) = 1$, $f(2) = 0$, $f(3) = -1$, and $f(4) = 0$. For $n \geq 5$ the function f is increasing, and $f(11) = 1927$ and $f(12) = 3952$. So $f(1)$, $f(5)$, $\ldots$, $f(11)$ correspond to stable positive integers less than 2017 of which there are $\boxed{8}$.

I-2 Compute the area of the quadrilateral with vertices at $(1,1)$, $(4,7)$, $(5,3)$, and $(2,0)$.

I-2 Solution Break the quadrilateral into two triangles, then the area is equal to

$$\frac{1}{2}\left(\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 5 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 5 & 3 \\ 1 & 4 & 7 \end{vmatrix}\right) = \frac{1}{2}(6+18) = \boxed{12}.$$

I-3 The sum of the squares of five consecutive positive integers, the largest integer being n, is equal to the sum of the squares of the next four consecutive integers. Compute n.

I-3 Solution The property stated in the problem translates into $(n-4)^2 + (n-3)^2 + (n-2)^2 + (n-1)^2 + n^2 = (n+1)^2 + (n+2)^2 + (n+3)^2 + (n+4)^2$. Pairing the $(n-k)^2$ and the $(n+k)^2$ terms on both sides of the equation, the n^2 and constant terms cancel, leaving $n^2 - 20n = 20n \to n^2 - 40n = 0 \to n = \boxed{40}$. Note that the case $n = 0$ is omitted as the five consecutive integers must be positive.

I-4 The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the sum of its coefficients is equal to the sum of its roots and is also equal to the product of its roots. If $P(0) = 4$, compute b.

I-4 Solution The sum of the coefficients of P is $1+a+b+c$, the sum of its roots is $-a$ and the product of its roots is $-c$. As $P(0) = 4$, it follows that $a = c = 4$ and $1+4+b+4 = -4 \to b = \boxed{-13}$.

I-5 $ABCDEFG$ is a pyramid with a hexagonal base. Compute the number of distinct ways all seven vertices of $ABCDEFG$ can be colored one of either red, blue, or green such that no vertices that share an edge are identically colored.

I-5 Solution Without loss of generality, let $ABCDEF$ be the base of the pyramid. Once a color is assigned to G, the colors of $ABCDEF$ must alternate between the two remaining colors. As there are three ways to pick the color for G and then two ways to assign colors to the remaining vertices, the answer is $\boxed{6}$.

I-6 In triangle ABC, if $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, compute the smallest possible value of $\cos C$.

I-6 Solution Noting that $\cos C = \cos(\pi-(A+B)) = \cos\pi\cos(A+B)+\sin\pi\sin(A+B) = -\cos(A+B)$, $-\cos(A+B) = -(\cos(A)\cos(B) - \sin(A)\sin(B))$. To minimize $\cos C$, $\cos(A)\cos(B)$ should be positive, so $\cos C = -\left(\frac{4}{5}\times\frac{12}{13} - \frac{3}{5}\times\frac{5}{13}\right) = \boxed{-\dfrac{33}{65}}$.

I-7 Compute $\log_2(3136) - \log_2(1764) + \log_2(900) - \log_2(400) + \log_2(144) - \log_2(36)$.

I-7 Solution It is hopefully observed that the terms in the alternating sum are of the form $(n(n+1))^2$ for $2 \le n \le 7$. The sum equals

$$\begin{aligned} 2\left(\log_2(56) - \log_2(42) + \log_2(30) - \log_2(20) + \log_2(12) - \log_2(6)\right) &= \\ 2\left((\log_2(8)+\log_2(7)) - (\log_2(7)+\log_2(6)) + (\log_2(6)+\log_2(5)) - \right. \\ \left.(\log_2(5)+\log_2(4)) + (\log_2(4)+\log_2(3)) - (\log_2(3)+\log_2(2))\right) &= \\ 2\left(\log_2(8)-\log_2(2)\right) &= \boxed{4}. \end{aligned}$$

I-8 The roots of $10x^2 - 14x + k$ are $\sin\alpha$ and $\cos\alpha$ for some real value of α. Compute k.

I-8 Solution Noting that $\sin\alpha + \cos\alpha = \frac{7}{5}$, $\frac{49}{25} = (\sin\alpha+\cos\alpha)^2 = (\sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha) = 1 + 2\left(\frac{k}{10}\right) \to k = \boxed{\dfrac{24}{5}}$.

I-9 Compute the rightmost non-zero digit in the base-8 expansion of 17!.

I-9 Solution The prime factorization of 17! is $(2)(3)(2^2)(5)(2\times3)(7)(2^3)(3^2)(2\times5)(11)(2^2\times3)(13)(2\times7)(3\times5)(2^4)(17) = 2^{15}\times3^6\times5^3\times7^2\times11\times13\times17$. Because of the 2^{15} term, 17! will have 5 terminal zeroes when written in base-8. Accordingly, it is necessary to determine the product of the remaining terms in the prime factorization modulo 8 to determine the rightmost non-zero digit. Each terms' remainder modulo 8 is, in order, 1, 5, 1, 3, 5, and 1, for a product of 75, which has a remainder of $\boxed{3}$ modulo 8. Note: $17! = 355687428096000_{10} = 12067735663300000_8$.

I-10 Let S be a set of 100 points inside a square of side length 1. An ordered pair of not necessarily distinct points (P,Q) is *bad* if $P \in S$, $Q \in S$ and $|PQ| < \frac{\sqrt{3}}{2}$. Compute the minimum possible number of bad ordered pairs in S.

I-10 Solution First, note that among any five points there is a bad pair. To see this, divide the unit square into four identical sub-squares, and observe that any two points in the same sub-square are separated by a distance of at most $1/\sqrt{2}$. Consider a graph with the vertices representing the 100 points of S and an edge between two vertices if they are *not* a bad pair. Because of the above observation, the graph contains no complete subgraph on five vertices (in other words, no set of five vertices has every vertex joined to every other vertex in the set). Consequently, by Turán's theorem, the graph that contains no complete subgraph of five vertices that maximizes the number of edges in the graph (in other words, minimizes the number of bad pairs) has the vertices partitioned into 4 groups of 25 and connects every vertex to every other vertex provided they are in different groups. This corresponds to a construction of S where each corner of the square contains 25 points of S, for example. For this arrangement, there are $4 \times 25^2 = \boxed{2500}$ bad ordered pairs, and this is the fewest possible.

Relay Round (6 minutes, 8 minutes, 10 minutes)

R1-1 If x and y are non-negative integers such that $x < y$ and $x!y! = 10!$, compute the maximum possible value of x.

R1-1 Solution Observing that $10 \times 9 \times 8 = 720 = 6!$, $10! = 6!7!$, so the answer is $\boxed{6}$.

R1-2 Let T = TNYWR. $HAIRNETS$ is an equilateral concave octagon of side length T. The interior angle measurements of H, I, N, and T are 60 degrees, and the interior angle measurements of A, R, E, and S are 210 degrees. Compute the area of $HAIRNETS$.

R1-2 Solution The area octagon $HAIRNETS$ consists of a square ($ARES$) and four equilateral triangles (SHA, AIR, RNE, and ETS). The total area is $T^2 + 4\frac{\sqrt{3}T^2}{4}$. As $T = 6$, the answer is $\boxed{36 + 36\sqrt{3}}$.

R2-1 The side lengths of a triangle are $\sqrt{20}$, $\sqrt{17}$, and x. Compute the greatest possible integer value of x.

R2-1 Solution By the triangle inequality, the sum of two of the side lengths cannot exceed the length of the third side. Accordingly, if x is the length of the longest side, it is bounded above by $\sqrt{20} + \sqrt{17}$. As both values are between 4 ($\sqrt{16}$) and 4.5 ($\sqrt{20.25}$), their sum is greater than 8 but less than 9, so the greatest possible integer value of x is $\boxed{8}$.

R2-2 Let T = TNYWR. The finite set S has exactly T distinct subsets each containing an even number of elements. Compute the number of elements in S.

R2-2 Solution The set S has $2^{|S|}$ subsets, of which $2^{|S|-1}$ contain an even number of elements. As $T = 8$, $|S| - 1 = 3 \to |S| = \boxed{4}$.

R2-3 Let T = TNYWR. Let S be the set of non-negative integers less than T^3. Compute the (base-10) sum of the digits of all of the elements of S written in base T.

R2-3 Solution Including trailing zeroes, the elements of S when written in base-T are $\{000_T, 001_T, \ldots, \underline{(T-1)}\ \underline{(T-1)}\ \underline{(T-1)}_T\}$. By symmetry, every digit between 0 and $T-1$ inclusive appears in each position T^2 times, so the sum of all of the digits is $3 \times T^2 \times \frac{(T-1)(T)}{2}$. As $T = 4$, the answer is $3 \times 16 \times 6 = \boxed{288}$.

R3-1 Compute the number of distinct ways to erase two of the decimal digits of 9876543210 and obtain an eight-digit number that is divisible by 9.

R3-1 Solution The given number has sum of digits 45, so two digits whose sum is divisible by 9 should be erased. There are $\boxed{5}$ ways to do this, namely $0+9$, $1+8$, $2+7$, $3+6$, and $4+5$.

R3-2 Let $T =$ TNYWR. ABC is an isosceles right triangle. If the longest median has length T, compute the area of ABC.

R3-2 Solution Let A be the right angle of the triangle. The longest median goes from one of the vertices of the hypotenuse (say C) to the opposite side (say a point D on $\overline{AB}$). For simplicity, let $AB = 2x$, then the area of ABC is $2x^2$. Additionally, $CD^2 = CA^2 + AD^2 = 5x^2$, so the area of the triangle is $\frac{2}{5}$ the square of the length of the longest median. As $T = 5$, the area is $\boxed{10}$.

R3-3 Let $T =$ TNYWR. The polynomial x^3+3x^2+px+T is evenly divisible by $x+2$. Compute p.

R3-3 Solution As -2 is a root of the polynomial, $(-2)^3 + 3(-2)^2 + p(-2) + T = 0 \to 2p = T + 4$. As $T = 10$, $p = \boxed{7}$.

R3-4 Let $T =$ TNYWR. If p and q are the roots of $x^2 + x + 3$, then $x^2 + bx + c$ has roots $p+T$ and $q+T$. Compute c.

R3-4 Solution Noting that $x^2+bx+c = (x-(p+T))(x-(q+T)) = x^2-(p+q+2T)x+(p+T)(q+T)$, then $c = (p + T)(q + T) = pq + (p + q)T + T^2$. The sum and product of the roots of x^2+x+3 are -1 and 3, respectively, so $c = T^2 - T + 3$. As $T = 7$, $c = 49 - 7 + 3 = \boxed{45}$.

R3-5 Let $T =$ TNYWR. Phil has a stack of money consisting of 5-dollar and 20-dollar bills worth 780 dollars in total. If there are T bills in total, compute the number of 5-dollar bills.

R3-5 Solution Let F be the number of 5-dollar bills. Then there are $T - F$ 20-dollar bills and $5F + 20(T - F) = 780$, so $F = \frac{20T-780}{15}$. As $T = 45$, $F = \frac{900-780}{15} = \boxed{8}$.

R3-6 Let $T =$ TNYWR. Compute the remainder when the eight-digit number $\underline{T}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{T}$ is divided by 13.

R3-6 Solution Let N be the number above, then $N = 10000001T+2342340 = 10000001T+234\times 1001\times 10$. As $1001 = 7 \times 11 \times 13$, $N \equiv 10000001T \pmod{13}$. As $10000001 \equiv 11 \pmod{13}$, $N \equiv 11T \pmod{13}$. As $T = 8$, $N \equiv 88 \equiv \boxed{10} \pmod{13}$.

Tiebreaker (10 minutes)

TB $ABCD$ is an isosceles trapezoid with $\overline{AD} \parallel \overline{BC}$ and $AD < BC$. E lies on $\overline{BC}$ such that $\overline{AE} \perp \overline{BC}$ and let M be the midpoint of $\overline{BC}$. Lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{AM}$ meet at G. Given that triangle GEM has area 20 and $AB = AM = 17$, compute the area of triangle ABC.

TB Solution First note that $AM = 3GM$ (in particular, G is the centroid of ABC). To see this, let F be the foot of the altitude from D. Then $AEFD$ is a rectangle and $EM = \frac{1}{2}AD$. So $\triangle GEM \sim \triangle GDA$ with ratio 2, hence $AG : GM = 2$ as claimed. Then, $[GEM] = 20$ implies $[AEM] = 60$. Since $AB = AM$, $[ABM] = 120$, and finally $[ABC] = 2[ABM] = \boxed{240}$.