2017 ARML Local Problems
Team Round (45 minutes)

T-1 A fair six-sided die has faces with values 0, 0, 1, 3, 6, and 10. Compute the smallest positive integer that *cannot* be the sum of four rolls of this die.

T-2 For a positive integer k, let z_k be the number of terminal zeroes of the product $1!2!\cdots k!$. For example, $z_6 = 2$ because $1!2!3!4!5!6! = 24883200$. Compute z_{100}.

T-3 Let $ARML$ be a square of side length 5. A point B on side $\overline{MR}$ and C on side $\overline{ML}$ are selected uniformly at random and independent of one another. Compute the expected area of triangle ABC.

T-4 The edges of regular hexagon $ABCDEF$ are made of mirrors. A laser is fired from A toward the interior of edge $\overline{CD}$, striking it at point G. The laser beam reflects off the interior of exactly one additional edge and returns to A. Compute $\tan(\angle DAG)$.

T-5 Let S be the set of lattice points $\{(x, y) : x, y \in \mathbb{Z}, 0 \leq x \leq 3, 0 \leq y \leq 4\}$. Compute the number of subsets of S of 4 lattice points that form the vertices of a square.

T-6 If A is an acute angle such that $\sin 15^\circ + \cos 15^\circ = \sqrt{2} \sin A$, compute $\cos A$.

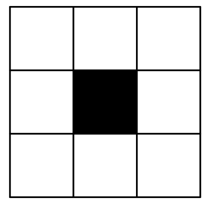
T-7 Consider the following algorithm for a given non-negative integer n:
Step 1: Initialize $k = 1$.
Step 2: Replace n with the product of its digits.
Step 3: If $n \leq 9$ return k and exit. If $n > 9$, increment k by 1 and return to Step 2.

For example, if $n = 125$, then the algorithm returns 2 because 125 is replaced by $1 \times 2 \times 5 = 10$ which is replaced by $1 \times 0 = 0$, at which point the algorithm returns 2 and exits. Compute the smallest value of n such that this algorithm returns 3.

T-8 Three co-planar squares, $BAHT$, $CAIN$, and $BCGY$ have areas 16, 16, and 32, respectively. If the squares only intersect pairwise at the vertices A, B, and C, compute the area of the convex hexagon $THINGY$.

T-9 Compute the coefficient of x^8 in the expansion of $(x^2 + x + 1)^8$ after combining like terms.

T-10 Consider the following 3×3 grid with its center square removed, as shown.



Compute the number of distinct ways to fill in the grid with the integers 1 through 8, each appearing exactly once, such that:

– In each of the three rows, the entries are increasing from left to right.
– In each of the three columns, the entries are increasing from top to bottom.

T-11 Kevin and Tarsha are playing a game with four fair standard six-sided dice. The four dice are rolled and if the numbers appearing on top of the four dice are all different, Kevin pays Tarsha \$1. If not, Tarsha pays Kevin \$$k$ dollars. Compute the value of k such that the game is fair, in other words, the expected value of each play of the game to both players is zero.

T-12 Compute the minimum value of ab such that $\log_2(a^4b^{-3}) = 3$ and $\log_2(a^4b^3) = 9$.

T-13 Compute

$$\sum_{n=3}^{10} \frac{20}{(n-2)(n+2)}$$

.

T-14 The six-digit number 142857 has the property that moving the rightmost digit to the left of the number results in multiplying the number by five ($142857 \times 5 = 714285$). Compute the smallest six-digit number with the property that moving the rightmost digit to the left of the number results in multiplying the number by four.

T-15 Compute the number of distinct sequences $(x_1, x_2, \ldots, x_{14})$ with the following properties:

– $x_k \in \{1, 2, \ldots, 14\}$ for each $k = 1, 2, \ldots, 14$.
– The number $20x_{k+1}^2 - 17x_k^2$ is divisible by 14 for each $k = 1, 2, \ldots, 13$.

Individual Round (10 minutes per pair)

I-1 A positive integer m is *stable* if $m = 2^n - n^2$ for some positive integer n. Compute the number of stable positive integers less than 2017.

I-2 Compute the area of the quadrilateral with vertices at $(1, 1)$, $(4, 7)$, $(5, 3)$, and $(2, 0)$.

I-3 The sum of the squares of five consecutive positive integers, the largest integer being n, is equal to the sum of the squares of the next four consecutive integers. Compute n.

I-4 The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the sum of its coefficients is equal to the sum of its roots and is also equal to the product of its roots. If $P(0) = 4$, compute b.

I-5 $ABCDEFG$ is a pyramid with a hexagonal base. Compute the number of distinct ways all seven vertices of $ABCDEFG$ can be colored one of either red, blue, or green such that no vertices that share an edge are identically colored.

I-6 In triangle ABC, if $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, compute the smallest possible value of $\cos C$.

I-7 Compute $\log_2(3136) - \log_2(1764) + \log_2(900) - \log_2(400) + \log_2(144) - \log_2(36)$.

I-8 The roots of $10x^2 - 14x + k$ are $\sin \alpha$ and $\cos \alpha$ for some real value of α. Compute k.

I-9 Compute the rightmost non-zero digit in the base-8 expansion of $17!$.

I-10 Let S be a set of 100 points inside a square of side length 1. An ordered pair of not necessarily distinct points (P, Q) is *bad* if $P \in S$, $Q \in S$ and $|PQ| < \frac{\sqrt{3}}{2}$. Compute the minimum possible number of bad ordered pairs in S.

Relay Round (6 minutes, 8 minutes, 10 minutes)

R1-1 If x and y are non-negative integers such that $x < y$ and $x!y! = 10!$, compute the maximum possible value of x.

R1-2 Let T = TNYWR. $HAIRNETS$ is an equilateral concave octagon of side length T. The interior angle measurements of H, I, N, and T are 60 degrees, and the interior angle measurements of A, R, E, and S are 210 degrees. Compute the area of $HAIRNETS$.

R2-1 The side lengths of a triangle are $\sqrt{20}$, $\sqrt{17}$, and x. Compute the greatest possible integer value of x.

R2-2 Let T = TNYWR. The finite set S has exactly T distinct subsets each containing an even number of elements. Compute the number of elements in S.

R2-3 Let T = TNYWR. Let S be the set of non-negative integers less than T^3. Compute the (base-10) sum of the digits of all of the elements of S written in base T.

R3-1 Compute the number of distinct ways to erase two of the decimal digits of 9876543210 and obtain an eight-digit number that is divisible by 9.

R3-2 Let T = TNYWR. ABC is an isosceles right triangle. If the longest median has length T, compute the area of ABC.

R3-3 Let T = TNYWR. The polynomial x^3+3x^2+px+T is evenly divisible by $x+2$. Compute p.

R3-4 Let T = TNYWR. If p and q are the roots of x^2+x+3, then x^2+bx+c has roots $p+T$ and $q+T$. Compute c.

R3-5 Let T = TNYWR. Phil has a stack of money consisting of 5-dollar and 20-dollar bills worth 780 dollars in total. If there are T bills in total, compute the number of 5 dollar bills.

R3-6 Let T = TNYWR. Compute the remainder when the eight-digit number $\underline{T}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{T}$ is divided by 13.

Tiebreaker (10 minutes)

TB $ABCD$ is an isosceles trapezoid with $\overline{AD} \parallel \overline{BC}$ and $AD < BC$. E lies on $\overline{BC}$ such that $\overline{AE} \perp \overline{BC}$ and let M be the midpoint of $\overline{BC}$. Lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{AM}$ meet at G. Given that triangle GEM has area 20 and $AB = AM = 17$, compute the area of triangle ABC.